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Eigenvarieties associated to Hilbert modular forms

by

Christopher Birkbeck

Thesis

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Contents

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Declarations

Part I of this thesis is entirely expository and none of the results contained therein are new. Chapters 3,4,5 are also mainly expository with some modifications due to the author. I declare that, unless otherwise indicated and to the best of my knowledge, the contents of this thesis is my own original work.

This thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy. No part of this thesis has been submitted towards any other degree.

Abstract

We use results by Chenevier and Hansen to interpolate the classical Jacquet-Langlands correspondence for Hilbert modular forms, which gives an extension of Chenevier's results to totally real fields. From this, in the case of totally real fields of even degree, we obtain isomorphisms between eigenvarieties attached to Hilbert modular forms and those attached to modular forms on a totally definite quaternion algebra. More generally, for any field and quaternion algebra, we get closed immersions between certain eigenvarieties associated to overconvergent cohomology groups.

Using this, we compute slopes of Hilbert modular forms near the centre and near the boundary of the weight space and prove a lower bound on the Newton polygon associated to the U_p operator. Near the boundary of the weight space we give evidence that the slopes are given by unions of arithmetic progressions and we give a conjectural recipe for generating slopes.

Introduction

History

The idea of modular forms living in p-adic families began with Serre [Ser73] who considered p-adic limits of compatible families of q-expansions of modular forms. This was the starting point for a vast theory. The work of Serre was then formalized by Katz [Kat73], who reformulated these ideas into a more geometric context and showed that the p-adic families were in fact part of a wider range of p-adic objects. After this, Dwork studied the action of the U_p operator on these spaces and showed it is a compact operator, thus giving us a way to study these spaces in much more detail. Then Hida, in a series of papers in the 1980's, showed that the space of p-ordinary eigenforms (which means that their U_p eigenvalue is a p-adic unit) of weight $k \geq 3$ has rank depending only on k modulo $p-1$ (or 2 for $p=2$). From this it follows that these p-ordinary modular forms naturally live in p-adic families. This was then extended by Coleman-Mazur and Buzzard [CM98, Buz07] to finite slope eigenforms (which means the U_p eigenvalue is not 0), by constructing geometric objects which they called eigencurves or more generally eigenvarieties. These are rigid analytic varieties which parametrize all such modular forms of a fixed level and their points correspond to systems of Hecke eigenvalues of finite slope overconvergent modular forms. In [Buz07], Buzzard creates an eigenvariety machine, which can be used to construct eigenvarieties by inputting a weight space and some suitable Banach modules together with an action of a Hecke algebra. Using this, Ash-Stevens [AS97], Hansen [Han] and Urban [Urb11] (among others) have used overconvergent cohomology groups to construct eigenvarieties associated to a large class of reductive groups. Eigenvarieties have many applications such as in the proof of the Fontaine-Mazur conjecture for GL_2. Moreover, understanding their geometry can give insights into parity conjectures for Selmer groups associated to modular forms (cf. [PX14]).

p-adic Langlands functoriality

Using the more general constructions of eigenvarieties given by [Buz07], it is possible to construct eigenvarieties associated to spaces of modular forms on a quaternion algebra D (so called quaternionic modular forms) and one can then study their relationship to eigenvarieties associated to spaces of modular forms (on GL_2). With this in mind, we recall that the classical Jacquet-Langlands correspondence tells us (roughly) that the classical spaces of quaternionic modular forms $S_k^D(N)$ of weight k and level N on a quaternion algebra D are isomorphic as Hecke modules to the spaces $S_k^{\mathfrak{d}\text{-new}}(N\mathfrak{d})$ of $\mathfrak{d}$-new modular forms for GL_2, where D has discriminant $\mathfrak{d}$. One can then ask if this extends to families of modular forms, i.e., if we can use this to relate the eigenvariety $\mathscr{X}_D$ coming from these quaternionic modular forms to the eigenvariety $\mathscr{X}_{\mathrm{GL}_2}$ coming from the usual spaces of modular forms . Over $\mathbb{Q}$, this was answered by Chenevier in [Che05], who showed that there is a closed immersion $\mathscr{X}_D \hookrightarrow \mathscr{X}_{\mathrm{GL}_2}$ which interpolates the classical Jacquet-Langlands correspondence[1]. This result is an instance of what is now called *p-adic Langlands functoriality*. Other examples of this can be found in work of Hansen [Han] and Ludwig [Lud14].

Going back to the result of Chenevier, if one picks $D/\mathbb{Q}$ to be a definite quaternion algebra, then the spaces of overconvergent quaternionic modular forms are of a combinatorial nature due to their much simpler geometry. This means that, if one is interested in computing the action of the U_p operator on the space of overconvergent modular forms, then one can reduce to computing on spaces of overconvergent quaternionic modular forms. This is the strategy used in [LWX14, WXZ14]. Our first goal is to extend the results of Chenevier to a totally real field F. In particular, we have the following theorem:

Theorem. *Let D/F be a totally definite quaternion algebra of discriminant $\mathfrak{d}$ defined over a totally real field F. Let p be a rational (unramified) prime and $\mathfrak{n}$ an integral ideal of F such that $p \nmid \mathfrak{n}\mathfrak{d}$ and $(\mathfrak{n}, \mathfrak{d}) = 1$. Let $\mathscr{X}_D(\mathfrak{n}p)$ be the eigenvariety of level $\mathfrak{n}p$ attached to quaternionic modular forms on D. Similarly, let $\mathscr{X}_{\mathrm{GL}_2}(\mathfrak{n}\mathfrak{d}p)$ denote the eigenvariety associated to cuspidal Hilbert modular forms of level $\mathfrak{n}\mathfrak{d}p$ (with the associated moduli problem for this level being representable) as constructed in [AIP16b].*

Then there is a closed immersion $\iota_D : \mathscr{X}_D(\mathfrak{n}p) \hookrightarrow \mathscr{X}_{\mathrm{GL}_2}(\mathfrak{n}\mathfrak{d}p)$ which interpolates the classical Jacquet-Langlands correspondence. Moreover, when $[F : \mathbb{Q}]$ is even, one can choose D with $\mathfrak{d} = 1$ so that the above is an isomorphism between the corresponding eigenvarieties.

More generally, for any quaternion algebra D and sufficiently small level (see Definition 1.3.6) $\mathfrak{n}p$, we construct (following [Han]) eigenvarieties $\mathscr{H}_D(\mathfrak{n}p)$ associated to

[1] In general Chenevier proves that one gets a isomorphism onto the $\mathfrak{d}$-new 'part' of the eigenvariety.

overconvergent cohomology groups $\mathrm{H}^\bullet(Y_D(\mathfrak{n}p), \mathcal{L}(\mathscr{D}_{\lambda_{\kappa,r}}))$, where $Y_D(\mathfrak{n}p)$ is a Shimura variety associated to D and $\mathcal{L}(\mathscr{D}_{\lambda_{\kappa,r}})$ is a local system on $Y_D(\mathfrak{n}p)$, where $\mathscr{D}_{\lambda_\kappa}$ is a distribution module (see Chapter 5 for the relevant definitions) . In this case, we use results of Hansen in [Han] to prove:

Theorem. *Let D be any quaternion algebra of discriminant $\mathfrak{d}$ and $\mathfrak{n}p$ a sufficiently small level. Let $\mathscr{H}_D(\mathfrak{n}p), \mathscr{H}_G(\mathfrak{n}p)$ be the eigenvarieties associated to overconvergent cohomology groups for $\mathrm{Res}_{F/\mathbb{Q}}(D^\times)$ and $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$ respectively. Then there is a closed immersion*

$$\mathscr{H}_D(\mathfrak{n}p)^o \hookrightarrow \mathscr{H}_G(\mathfrak{n}p\mathfrak{d})$$

interpolating the classical Jacquet-Langlands correspondence. Here $\mathscr{H}_D(\mathfrak{n}p)^o$ denotes the core of the eigenvariety $\mathscr{H}_D(\mathfrak{n}p)$ (see Definition 2.5.8).

Slopes of Hilbert modular forms

Our second goal is to use the overconvergent Jacquet-Langlands correspondence to study the p-adic valuation of the U_p eigenvalues (called the *slopes*). In the case of modular forms over $\mathbb{Q}$ this question has received a lot of attention recently, with a focus on studying slopes of overconvergent modular forms as they move in p-adic families. To make this more precise, consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ and let $\mathcal{W}$ be the associated rigid analytic space, which is called the *weight space*. Elements of $\mathcal{W}(\mathbb{C}_p)$ are identified with continuous homomorphisms $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, which are called *weights*. If we write $\mathbb{Z}_p^\times \cong H \times (1+q\mathbb{Z}_p)$ where H is the torsion subgroup and where $q = p$ if p is odd and $q = 4$ for $p = 2$, then taking a primitive Dirichlet character ψ modulo p^t and the character z^k of $1+q\mathbb{Z}_p$ sending $z \mapsto z^k$ for $k \in \mathbb{Z}$, we get an element of the weight space given by $z^k\psi$. The weights of the form z^k are called *algebraic* and weights of the form $z^k\psi$ are called *arithmetic*. If we now take γ a fixed topological generator of $1+q\mathbb{Z}_p$, and let $w(\kappa) = \kappa(\gamma) - 1$ for κ a weight, then the algebraic weights are in the region of the weight space such that $\mathrm{val}_p(w(\kappa)) \geq 1$ (for p odd) called the centre, and the arithmetic weights $z^k\psi$ (for ψ sufficiently ramified at p) are on the boundary where $\mathrm{val}_p(w(\kappa)) \leq \frac{1}{p-1}$ (again for p odd[2]). The reason we make such a distinction is that the behaviour of the slopes of the U_p operator acting on weight κ modular forms depends on where in the weight space κ lives, as we shall see later. Lastly, we note that $\mathcal{W} \cong \bigsqcup_\chi \mathcal{W}_\chi$ where the χ run over characters of H and $\mathcal{W}_\chi$ is the corresponding component of the weight space.

Over $\mathbb{Q}$, the behaviour of the slopes of U_p was first studied by Gouvéa-Mazur in [GM92] where they conjectured that if k_1, k_2 are large enough with $k_1 \equiv k_2 \mod p^n(p-1)$

[2]For $p = 2$ the centre is where $\mathrm{val}_2(-) \geq 3$ and the boundary where $\mathrm{val}_2(-) < 3$.

for $n \geq \alpha$ for some rational number α, then the dimension of the space of modular forms of weight k_1 and slope α should be the same as that of weight k_2 and slope α. Inspired by this, Buzzard, Calegari, Jacobs, Kilford and Roe (among others) computed and studied slopes of modular forms for weights both in the centre and boundary of the weight space. In particular, in [Buz05], Buzzard computed slopes in many cases and was able to make precise conjectures about their behaviour. Very little is known about the slopes near the centre of weight space and the geometry of the eigenvariety is expected to be more complicated. Results about slopes in this case can be found in [BC05, BP16b]. In particular, Bergdall-Pollack have constructed a 'ghost series' which conjecturally explains much of the behaviour of the slopes both near the centre and boundary of the weight space.

Near the boundary Buzzard–Kilford, Jacobs and Roe were among the first to give evidence that the sequence of slopes appear as a union of arithmetic sequences with same common difference. This then implies that over the boundary of the weight space the eigenvariety looks like a countable union of annuli. For $p = 2, 3$ and trivial tame level this was proven by Buzzard-Kilford and Roe in [BK05, Roe14]. For more details on the precise conjectures and their implications, see [BGar]. More generally, the recent work of Liu-Wan-Xiao and Wan-Xiao-Zhang in [LWX14, WXZ14] have proven similar results by working with quaternion algebras and using Chenevier's results mentioned above. In particular, they have defined 'integral models' for these spaces of modular forms and from this shown that over the boundary of weight space the eigenvariety associated to a totally definite quaternion algebra over $\mathbb{Q}$ is the disjoint union of countably many annuli. The existence and construction of these integral models is a very active area of research (see for example [AIP, AIP16a, BP16a, BP16b, BGar, JN16]). Understanding the geometry of eigenvarieties has many number theoretical applications; for example Pottharst and Xiao in [PX14] have recently reduced the parity conjecture of Selmer ranks for modular forms to a similar statement about the geometry of the eigenvariety.

In general, for overconvergent modular forms over $\mathbb{Q}$ we have the following conjecture (which can be found in [LWX14, BP16a]) for the behaviour of the Newton polygon of U_p.

Conjecture. (Folklore) *For κ a weight, let $s_1(\kappa), s_2(\kappa), \ldots$ denote the slopes of the Newton polygon of U_p acting on the spaces of overconvergent modular forms of weight κ and fixed level. Let $\mathrm{NP}_\kappa(U_p)$ be the Newton polygon of $\det(1 - XU_p)$. Then there exists an $r > 0$ depending only on the component $\mathcal{W}_\chi$ of the weight space containing κ, such that*

(a) *For $\kappa \in \mathcal{W}_\chi$ with $0 < \mathrm{val}_p(w(\kappa)) < r$, $\mathrm{NP}_\kappa(U_p)$ depends only on $\mathrm{val}_p(w(\kappa))$. Moreover, for weights in this component, the break points of the Newton polygon are independent of κ.*

(b) *The sequence* $\{s_i(\kappa)/\operatorname{val}_p(w(\kappa))\}$ *is a finite union of arithmetic progressions (after possibly removing a finite number of terms), which is independent of* κ *for* $0 < \operatorname{val}_p(w(\kappa)) < r$.

(c) *Assuming* (a) *above, the set of slopes* $s_i(\kappa)$ *are given by*

$$\bigcup_{i=0}^{\infty}\left(S_{seed} + i \cdot \frac{|\,H\,|}{2}\right),$$

where S_{seed} *is a fixed finite set*[3]*, which only depends on the number of cusps of* $X_0(M)$ *(with* M *the tame level) and the classical slopes in weight* 2 *at different components of the weight space.*[4]

Our goal here is to give computational evidence for a similar structure to the slopes of overconvergent Hilbert modular forms (in particular part (c) above) and prove a lower bound for the Newton polygon of U_p. We compute explicit examples of sequences of slopes of the U_p operator by using the overconvergent Jacquet-Langlands correspondence. Throughout, we work with arithmetic weights both in the centre and boundary of the weight space. The reason we only do this for arithmetic weights is for simplicity and these results can most certainly be extended to any weight.

Our computations show that, for κ near the boundary of the weight space (see Definition 3.4.12), the slopes of classical Hilbert modular forms in weight κ are given by unions of arithmetic progression. Our methods also allow us to compute finite approximations $U_p(N,\kappa)$ to the infinite matrix of U_p acting on overconvergent Hilbert modular forms of weight κ. In this case, since the U_p operator is compact, one can prove there exists a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ (see Warning 8.0.2 for an explicit lower bound of this function) such that if the size of our approximation matrix is $N \times N$, then the first $f(N)$ smallest slopes of $U_p(N,\kappa)$ coincide with the first $f(N)$ smallest slopes of overconvergent Hilbert modular forms of weight κ. Unfortunately, the best bounds on f that we have grow very slowly as N increases; this means that, in practice, to prove that all of the approximated slopes we have computed are in fact slopes of overconvergent Hilbert modular forms (which we expect is the case), our N needs to be much larger than we can currently compute with.[5]

Our computations do however have much of the (conjectural) structure that one has over $\mathbb{Q}$; meaning there is evidence that the overconvergent slopes can be 'generated'

[3] Here the notation is such that if S is a set of slopes and $i \in \mathbb{Z}$, then we let $S + i$ denote the set, where we add i to each slope in S.

[4] This was shown to follow from (a) by Bergdall-Pollack in [BP16a].

[5] For example, in some of our computations, we would need our approximation matrix to have $N \sim 10^6$, although computations suggest that, in this case, we only need $N \sim 480$, but we cannot at this time prove this much stronger bound.

by slopes appearing in the classical spaces of Hilbert modular forms of (parallel) weight 2 analogous to what one sees over $\mathbb{Q}$ (e.g. part (c) of the conjecture above). See Conjecture 8.3.1. As an example of the computations we have done, we have the following:

Example (Split case). Let $F = \mathbb{Q}(\sqrt{13})$, $p = 3$ (which is split), level $U_0(9)$ and nebentypus ψ of conductor 9. Then we have the following sequence of approximated slopes of U_p (here (and throughout) we write (s, m) for the slope s together with the multiplicity m with which it appears. The size corresponds to the size of our approximation matrix $U_p(N, \kappa)$).

Weight	Size	Slopes
$[4,4]\mathbb{1}$	$20 \cdot 12$	$(0,1)$, $(1,2)$, $(2,4)$, $(3,4)$, $(4,6)$, $(5,10)$, $(6,7)$, $(7,6)$, $(15/2,2)$, $(8,12)$, $(17/2,2)$, $(9,5)$,$(\infty,144)$
$[4,4]\psi$	$30 \cdot 12$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,40)$, $(11/2,22)$, $(6,48)$, $(13/2,26)$, $(7,50)$, $(15/2,18)$, $(8,19)$, $(17/2,4)$, $(9,2)$

Observation. (a) Looking at these computations we see that near the centre of the weight space, where the character is trivial ($\mathbb{1}$), the slopes are 'almost' in arithmetic progression apart from a few entries, but there is little structure to the multiplicities. Moreover, in this case, since the wild level is large, we see that we get lots of forms of infinite slope, as is expected.

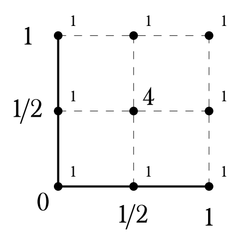
(b) If we now pick ψ to be a character of conductor 9, and consider weight $[4,4]\psi$, then the sequence of approximated slopes appears as the union of arithmetic progressions with common difference $1/2$. Moreover, we see that the multiplicities with which the slopes appear is increasing, which is something that cannot happen (in many cases) for modular forms over $\mathbb{Q}$ by the work of [LWX14]. In Chapter 8 we give some insight as to why we see this phenomenon and observe some structure of the slopes similar to that in [BP16a]. Lastly, we will see that the computations indicate that these slopes (once appropriately normalized) only depend on which component of the weight space the weight lies (and possibly on how close to the boundary of the weight space we are).

Since we are in the split case, we have that $U_p = U_{\mathfrak{p}_1} U_{\mathfrak{p}_2}$ where $U_{\mathfrak{p}_i}$ are commuting Hecke operators, therefore one can study the slopes of $U_{\mathfrak{p}_i}$ acting on classical spaces of Hilbert modular forms in order to understand the classical slopes of U_p. To this end we have the following table listing classical slopes of $U_p, U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}$.

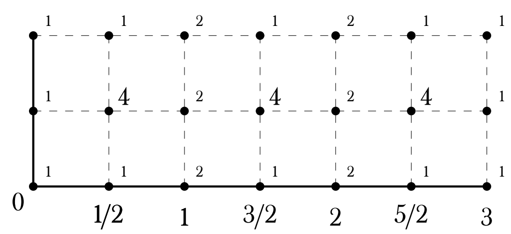
Operator	Weight	Classical Slopes
U_p	$[2,2]\psi$	$(0,1)$, $(1/2,2)$, $(1,6)$, $(3/2,2)$, $(2,1)$
$U_{\mathfrak{p}_1}$	$[2,2]\psi$	$(0,3)$, $(1/2,6)$, $(1,3)$
$U_{\mathfrak{p}_2}$	$[2,2]\psi$	$(0,3)$, $(1/2,6)$, $(1,3)$
U_p	$[4,2]\psi$	$(0,1)$, $(1/2,2)$, $(1,7)$, $(3/2,4)$, $(2,8)$, $(5/2,4)$, $(3,7)$, $(7/2,2)$, $(4,1)$
$U_{\mathfrak{p}_1}$	$[4,2]\psi$	$(0,3)$, $(1/2,6)$, $(1,6)$, $(3/2,6)$, $(2,6)$, $(5/2,6)$, $(3,3)$
$U_{\mathfrak{p}_2}$	$[4,2]\psi$	$(0,9)$, $(1/2,18)$, $(1,9)$

In these cases one can write the slopes of U_p as a pair $(\lambda_{\mathfrak{p}_1}, \lambda_{\mathfrak{p}_2})$ where $\lambda_{\mathfrak{p}_i}$ is a slope of $U_{\mathfrak{p}_i}$ and $\lambda_p = \lambda_{\mathfrak{p}_1} + \lambda_{\mathfrak{p}_2}$ is the corresponding slope of U_p. This can be represented pictorially as:

Weight [2,2]



Weight [4,2]



where the axes represent the slopes of $U_{\mathfrak{p}_i}$ and the numbers on the grid are the multiplicity with which the pair of slopes appears as a slope of U_p. These computations suggest that, not only are the slopes of $U_{\mathfrak{p}_i}$ given as unions of arithmetic progressions, but that the above pictures have a precise structure independent of the weight. See Section 8.1 for more details and conjectures.

In the overconvergent case, we run into the problem that the action of $U_{\mathfrak{p}_i}$ on $S^{\dagger}_{\kappa,r}(U)$ is ***not*** compact, so one cannot directly compute their slopes (apart from the slopes of classical forms). To get around this, in Subsection 8.1.5 we describe two methods of computing overconvergent slopes of $U_{\mathfrak{p}_i}$ and give some examples.

Example (Inert case). Let $F = \mathbb{Q}(\sqrt{5})$, $p = 2$ (which is inert) level $8\mathfrak{p}_{11}$ (where $\mathfrak{p}_{11} \mid 11$) and a primitive Hecke character ψ of conductor 8. Then we have the following sequence of approximated slopes (with the same notation as above):

Level	Weight	Size	Slopes
$U_0(8\mathfrak{p}_{11})$	$[4,4]\psi$	$24 \cdot 16$	(2/3, 6), (1 , 4), (4/3, 6), (3/2, 8), (2, 16), (5/2, 16), (8/3, 6), (3, 20), (10/3, 6), (7/2, 16), (11/3, 12), (4, 24), (13/3, 12), (9/2, 24), (14/3, 6), (5, 36), (16/3, 6), (11/2, 24), (17/3, 12), (6, 24), (19/3, 12), (13/2, 16), (20/3, 6), (7, 20), (22/3, 6), (15/2, 16), (8, 16), (17/2 , 8)
$U_0(2\mathfrak{p}_{11})$	$[4,4]\mathbb{1}$	200	(2, 8), (4, 1), (5, 4), (16/3, 6), (6, 6), (7, 2), (8, 7), (17/2, 12), (9, 4), (10, 21), (11, 2), (12, 6), (25/2, 4), (13, 6), (27/2, 4), (14, 3), (15, 22), (16, 8), (33/2, 4), (17, 13), (35/2, 24), (18, 21), (19, 6), (39/2, 2), (20, 3), (21, 1)

Observation. (a) Here again we see that the approximated slopes are in arithmetic progression and again the multiplicities are increasing. We note here that in the example computed above, not all the arithmetic progressions have the same difference, which again is something that has not been observed in the case of modular forms over $\mathbb{Q}$ and again, by [LWX14], cannot happen in many cases. Lastly, as in the previous example, we will see in Chapter 8 that computations suggest that these slopes (once normalized) only depend on which component of the weight space the weight lives. See Conjecture 8.3.1.

(b) To contrast, we also compute slopes near the centre at a low level. Here we see much less structure and that many slopes are not integers. We note also that that 2 is $U_0(\mathfrak{p}_{11})$-irregular (cf. Section 8.4).

For D totally definite with $\mathrm{Disc}(D) = 1$ and U a sufficiently small level, let h be the number of points[6] in the corresponding Shimura variety $Y_D(U)$, (which is finite by Proposition 1.3.5). In Chapter 7, we will show that the U_p operator naturally has the form of an infinite block matrix whose blocks have size $h \times h$. Furthermore, we will see that the entries of the block matrices lying on the diagonal of U_p are given (up to a p-power factor) by uniformly continuous functions. Using this, we will give a criterion (which can be checked in finite time[7]) such that the submatrix of U_p given by deleting the lower diagonal blocks has slopes matching the computed slopes of U_p.

Remark. In all the cases we have computed, one conjectures (cf. Conjecture 8.3.1) that the slopes near the boundary can be generated by an algorithm whose only input is

[6] If we let $S_2^D(U)$ denote the space of weight $[2,2]$ modular forms on D including the space of elements that factor through the reduced norm map (these correspond to Eisenstein series, see [DV13, Definition 3.7]) then $h = \dim(S_2^D(U))$.

[7] Although, in our case, the check would take too long to finish, so we only check on a small subset.

the number of cusps and the slopes appearing in weight $[2,2]\psi\tau^n$ for τ the Teichmüller character, which is analogous to the folklore conjecture above for modular forms over $\mathbb{Q}$.

Remark. Our computations near the boundary, for a fixed field F and prime p, are limited to only changing the algebraic part of the weight and not the finite part, which means $\mathrm{val}_p(w(\kappa))$ (which is defined in 3.4.12) is always fixed. The reason for this is that changing $\mathrm{val}_p(w(\kappa))$ requires working with more ramified characters and levels, which translates into much larger matrices.

In Section 8.4, we also collect some computations of slopes for weight lying on the centre of the weight space (i.e. with trivial character). In this setting, we observe that near the centre the slopes are no longer given by unions of arithmetic progressions and that there is a more complicated structure to the slopes, which is analogous to the behaviour of modular forms over $\mathbb{Q}$.

Lastly, we prove a lower bound for the Newton polygon of U_p on overconvergent Hilbert modular forms over a real quadratic field (although this can easily be adapted for more general totally real fields of even degree).

Proposition. *Let D/F be totally definite with $\mathfrak{d} = 1$ and let U be a sufficiently small level. Let h be the class number of (D, U) (as defined in 1.3.3) and let κ be any arithmetic weight. Then the Newton Polygon of the action of U_p on overconvergent Hilbert modular forms of level U weight κ lies above the polygon with vertices*

$$(0,0),(h,0),(3h,2h),\ldots,\left(\frac{i(i+1)h}{2},\frac{(i-1)i(i+1)h}{3}\right),\ldots.$$

Remark. Note this this is simply the polygon with h slopes 0, $2h$ slopes 1, $3h$ slopes 2 and so on.

Notation

We collect here some of the most used notations throughout.

(1) Let F be a totally real field of degree g with ring of integers $\mathcal{O}_F$ and let $\mathfrak{d}_F$ denote the different ideal of F. Let Σ be the set of all places of F and $\Sigma_\infty \subset \Sigma$ the set of all infinite places of F.

(2) For each finite place v of F, let F_{v} (or $F_{\mathfrak{p}}$ for $\mathfrak{p}$ the corresponding prime ideal) denote the completion of F with respect to v and $\mathcal{O}_{\mathrm{v}}$ the ring of integers of F_{v}. For an integral ideal $\mathfrak{n}$, let $F_{\mathfrak{n}} = \bigoplus_{\mathrm{v}|\mathfrak{n}} F_{\mathrm{v}}$ and similarly let $\mathcal{O}_{\mathfrak{n}} = \bigoplus_{\mathrm{v}|\mathfrak{n}} \mathcal{O}_{\mathrm{v}}$. In particular, if we have $p\mathcal{O}_F = \prod_{i=1}^{f} \mathfrak{p}_i$, then let $\mathcal{O}_p = \bigoplus_i \mathcal{O}_{\mathfrak{p}_i} = \mathcal{O}_F \otimes \mathbb{Z}_p$.

(3) Let p be a rational prime which, unless otherwise stated, will be unramified in F. Let Σ_p be the set of primes/places dividing p in F. For each $\mathfrak{p} \in \Sigma_p$, we let $\pi_{\mathfrak{p}}$ denote a chosen (and fixed throughout) uniformizer of $F_{\mathfrak{p}}$ (the completion at F at $\mathfrak{p}$).

(4) Let $\mathbb{A}_F$ denote the adeles of F and $\mathbb{A}_{F,f}$ the finite adeles. In the case $F = \mathbb{Q}$ we drop the subscript F.

(5) For a fractional ideal $\mathfrak{r}$, let $\mathfrak{r}^+$ denote the totally positive elements in $\mathfrak{r}$, and in general '+' will denote 'totally positive'. Moreover, let $\mathfrak{r}^*$ denote $\mathfrak{r}^{-1}\mathfrak{d}_F^{-1}$. (Note that this means we have a pairing $\mathrm{Tr}_{F/\mathbb{Q}} : \mathfrak{r} \times \mathfrak{r}^* \to \mathbb{Z}$).

(6) Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ and we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Furthermore, we fix embeddings $inc : \overline{\mathbb{Q}} \to \mathbb{C}$ and $inc_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, which allow us to think of the elements of $\overline{\mathbb{Q}}$ as both complex and p-adic numbers.

(7) Let L be a complete extension of $\mathbb{Q}_p$, which contains the compositum of the images of F under $\iota \circ \iota_{\mathrm{v}}$, for $\mathrm{v} \in \Sigma_\infty$ where $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ such that $\iota \circ inc = inc_p$ and ι_{v} is the field embedding of F into $\mathbb{C}$ given by v.

(8) Let D be a quaternion algebra over F and let $G_D = \mathrm{Res}_{F/\mathbb{Q}}(D^\times)$ (we will sometimes abuse notation and denote this simply by D). When $D = M_2(F)$ we denote this simply as G. Let T denote a fixed maximal torus of G_D and $\mathbb{T} = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathbb{G}_m$.

Part I

Background

In this expository section, we recall definitions of the classical spaces of modular forms attached to a quaternion algebra over a totally real field, together with some background on the construction of eigenvarieties.

We begin with some general background of quaternion algebras and then (following Hida) define the relevant spaces of modular forms. We then state the classical Jacquet-Langlands correspondence which will be essential later on. We also review some of the elements used in the construction of eigenvarieties together with some of their properties. Lastly, we state Chenevier's Interpolation Theorem which will be used to relate different eigenvarieties.

Chapter 1

Classical background

1.1 Quaternion algebras

In this section we give a short introduction to quaternion algebras, including some classification results. These results are all classical and can be found in many places (see for example [Vig80]).

Definition 1.1.1. Let K be a field. A ***quaternion algebra*** over K is a K-algebra D, such that:

(a) the centre of D is exactly K;

(b) the dimension of D as a vector space over K is 4;

(c) D has no non-trivial 2-sided ideals.

Example 1.1.2. Let K be a field of characteristic different from 2 and let $\{i, j, k\}$ be such that $i^2 = a, j^2 = b, ij = k = -ji$ for $a, b \in K^\times$. A simple check shows that this defines a quaternion algebra, which we will denote by $\left(\frac{a,b}{K}\right)$.

Proposition 1.1.3. *If* $\mathrm{Char}(K) \neq 2$*, then any quaternion algebra can be written as* $\left(\frac{a,b}{K}\right)$ *for some* $a, b \in K^\times$.

Proof. See [Vig80, Chapter I]. $\square$

Example 1.1.4. If we take $K = \mathbb{R}$, then we have $M_2(\mathbb{R}) \cong \left(\frac{1,1}{\mathbb{R}}\right)$ and the Hamilton quaternions $\mathbb{H} \cong \left(\frac{-1,-1}{\mathbb{R}}\right)$. Moreover, one can show that up to isomorphism these are the only quaternion algebras over $\mathbb{R}$.

Definition 1.1.5. We say that a quaternion algebra over a field K is *split*, if it is isomorphic to $M_2(K)$, otherwise we say it is *ramified*. Furthermore, if L/K is a field extension, then $D \otimes_K L$ is a quaternion algebra over L and if this new quaternion algebra over L is split then we say that L *splits* D.

Proposition 1.1.6. *If $a \in K^\times$ is a square, then $\left(\frac{a,b}{K}\right) \cong M_2(K)$.*

Proof. Let $a = \alpha^2$, then the map $i \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)$, $j \mapsto \left(\begin{smallmatrix} 0 & 1 \\ b & 0 \end{smallmatrix}\right)$, $k \mapsto \left(\begin{smallmatrix} 0 & \alpha \\ -b\alpha & 0 \end{smallmatrix}\right)$ extends K-linearly to give an isomorphism. □

Corollary 1.1.7. *If K is an algebraically closed field, then the only quaternion algebra (up to isomorphism) is $M_2(K)$.*

In general, with slightly more work, one can prove the following:

Proposition 1.1.8. *If K is a finite extension of $\mathbb{Q}_p$, then there is a unique (up to isomorphism) division quaternion algebra.*

Proof. See [Vig80, Chapter II, Theorem 1.1]. □

Definition 1.1.9. If D is a quaternion algebra then there is a conjugation map $x \mapsto \bar{x}$. Using this, one defines the *reduced norm* $\mathrm{nrd} : D \to F^\times$ by $\mathrm{nrd}(x) = x\bar{x}$. Explicitly, for $D = \left(\frac{a,b}{F}\right)$ and $\alpha = u + vi + wj + zj \in D$, we have[1] $\mathrm{nrd}(\alpha) = u^2 - av^2 - bw^2 + abz^2$.

Definition 1.1.10. Let K be a number field and let D be a quaternion algebra over K. Let K_v be the completion of K at a place v and let $D_\mathrm{v} = D \otimes_K K_\mathrm{v}$. Then this new quaternion algebra can be split, in which case we say that D is *split at* v, otherwise we say D is *ramified at* v. Let $\mathrm{Ram}(D)$ denote the set of places at which D is ramified.

Definition 1.1.11. The *discriminant* $\mathrm{Disc}(D)$ of a quaternion algebra is the product of the *finite primes*[2] in $\mathrm{Ram}(D)$.

Theorem 1.1.12. *Let D, $\mathrm{Ram}(D)$ and K be as above, then:*

(1) The set $\mathrm{Ram}(D)$ is finite and has an even number of elements, none of which is complex.

(2) For any set S of places of K not containing the complex places and having an even number of elements, then there is exactly one quaternion algebra (up to isomorphism) D/K with $\mathrm{Ram}(D) = S$.

[1] When our quaternion algebra is $M_2(F)$ the reduced norm map is just the determinant map.
[2] Some authors also include the infinite places, but we do not follow this convention.

Proof. The proof of (1) comes down to using the product formula for Hilbert symbols, but (2) we requires a little more work. Both these results can be deduced from [Wei74, XIII.3, Theorem 2 and XIII.6, Theorem 4]. □

Definition 1.1.13. We say a quaternion algebra $D/\mathbb{Q}$ is ***definite*** if $\infty \in \mathrm{Ram}(D)$. In general, for a quaternion algebra D over a totally real field F, we say that D is ***totally definite*** if it ramifies at all the real places of F.

Definition 1.1.14. Let D/K quaternion algebra. A finitely generated $\mathcal{O}_K$-submodule I of D is called a $\mathcal{O}_K$***-lattice*** if $KI = D$. A $\mathcal{O}_K$-lattice is called an ***order*** if it is also a subring D. Lastly, an order is called ***maximal*** (resp. ***an Eichler order***) if it is not properly contained in any other order (resp. if it can be written as the intersection of two maximal orders).

1.2 Modular forms on quaternion algebras

In this section we define the spaces of quaternionic modular forms over a totally real field, following [Hid88].

Notation 1.2.1. (1) Let D/F denote a quaternion algebra over F with a fixed maximal order $\mathcal{O}_D$.

(2) For D a quaternion algebra, we set $\Sigma_D = \{\mathrm{v} \in \Sigma_\infty \mid D \otimes_F F_\mathrm{v} \cong M_2(F_\mathrm{v})\}$ and $\Sigma^D = \Sigma_\infty - \Sigma_D$. Note that Σ_D consists of all the infinite places where the quaternion algebra splits and Σ^D are all the infinite places where it ramifies.

(3) For $m \in \mathbb{Z}^{\Sigma_\infty}$, we set $m^D = (m_\mathrm{v})_{\mathrm{v}\in\Sigma^D}$ (also define m_D analogously with Σ_D in place of Σ^D).

1.2.2. Let $K_0/\mathbb{Q}$ (in $\mathbb{C}$) be a finite Galois extension with ring of integers $\mathcal{O}_{K_0}$ such that there is an isomorphism

$$\alpha_D : D \otimes_\mathbb{Q} K_0 \xrightarrow{\sim} M_2(K_0)^{\Sigma_\infty}$$

Moreover, we assume that under this isomorphism, the projection $D \to M_2(K_0)$ at each place $\mathrm{v} \in \Sigma_D$ sends D into $M_2(K_0 \cap \mathbb{R})$ and such that $\mathcal{O}_D \otimes_\mathbb{Z} \mathcal{O}_{K_0}$ is sent into $M_2(\mathcal{O}_{K_0})^{\Sigma_\infty}$ (such a K_0 always exists). Let $G_{D,\infty}$ denote the infinite part of $G_D(\mathbb{A})$. Then α_D induces an identification

$$G_{D,\infty} = \mathrm{GL}_2(\mathbb{R})^{\Sigma_D} \times (\mathbb{H}^\times)^{\Sigma^D},$$

where $\mathbb{H}$ is the usual Hamilton quaternion algebra. Let $C_\infty = (\mathbb{R}^\times O_2(\mathbb{R}))^{\Sigma_D} \times (\mathbb{H}^\times)^{\Sigma^D}$ and $C_\infty^+ = (\mathbb{R}^\times SO_2(\mathbb{R}))^{\Sigma_D} \times (\mathbb{H}^\times)^{\Sigma^D}$, where O_2 are the orthogonal matrices, and SO_2 are the special orthogonal matrices.

Definition 1.2.3. Let $t = |\Sigma^D|$. For A a $\mathcal{O}_{K_0}$-algebra and $n, v \in \mathbb{Z}_{\geq 0}^t$ we take

$$V(n, v, A) \subset A[Z_{\mathrm{v}_1}, \dots, Z_{\mathrm{v}_t}]$$

to be the space polynomials in variables $(Z_\mathrm{v})_{\mathrm{v}\in\Sigma^D}$ where the degree of Z_v is at most n_v. We give this space a right $\mathcal{O}_D$-action by letting $g \in \mathcal{O}_D$ with $\alpha_D(g) = \gamma = (\gamma_\mathrm{v})_\mathrm{v} \in M_2(\mathcal{O}_{K_0})^{\Sigma_\infty}$ act by:

$$g : \prod_{\mathrm{v}\in\Sigma^D} Z_\mathrm{v}^{m_\mathrm{v}} \longmapsto \prod_{\mathrm{v}\in\Sigma^D} (c_\mathrm{v} Z_\mathrm{v} + d_\mathrm{v})^{n_\mathrm{v}} \operatorname{nrd}(g)^{v_\mathrm{v}} \left(\frac{a_\mathrm{v} Z_\mathrm{v} + b_\mathrm{v}}{c_\mathrm{v} Z_\mathrm{v} + d_\mathrm{v}} \right)^{m_\mathrm{v}}$$

where $\gamma_\mathrm{v} = \begin{pmatrix} a_\mathrm{v} & b_\mathrm{v} \\ c_\mathrm{v} & d_\mathrm{v} \end{pmatrix}$ and then extend this action A-linearly. Note that $m_\mathrm{v} \leq n_\mathrm{v}$. For $f \in V(n, v, A)$ we denote this action by $f|_{n,v} r$, or simply $f|r$ if there is no risk of confusion. We denote the module by $V(n, v, A)$ or $V_{n,v}(A)$, but we note that this module depends on the splitting behaviour of D which we suppress in our notation. We note that this action naturally and uniquely extends to an action of $G_D(A)$.

Remark 1.2.4. Note that we have given a right action 'at infinity', meaning that the action is coming from splitting our quaternion algebra at infinity. Later, in the p-adic setting, we will work with totally definite quaternion algebras which we assume are split at all primes over p, in this case we will give an action at p.

Definition 1.2.5. If $x \in G_{D,\infty}$ and $J \subset \Sigma_D$, then we define a right action of $G_{D,\infty}$ on subsets Σ_D by setting $J^x = \{\mathrm{v} \in \Sigma_D \mid \mathrm{v} \in J \text{ and } \operatorname{nrd}(x_\mathrm{v}) > 0 \text{ or } \mathrm{v} \in \Sigma_D - J \text{ and } \operatorname{nrd}(x_\mathrm{v}) < 0\}$.

Note that given any J one can find $x_J \in C_\infty$ such that $J^{x_J} = \Sigma_D$. This can be done by setting x_J to be such that $\operatorname{nrd}(x_\mathrm{v}) > 0$ for $\mathrm{v} \in J$ and $\operatorname{nrd}(x_\mathrm{v}) < 0$ otherwise.

Definition 1.2.6. Let $\mathcal{H}$ be the complex upper half space and let $\mathcal{H}^{\Sigma_D}$ be $|\Sigma_D|$-copies of $\mathcal{H}$ indexed by the elements of Σ_D. For each subset $J \subset \Sigma_D$ define the ***automorphy factor***

$$j_J : G_{D,\infty} \times \mathcal{H}^{\Sigma_D} \longrightarrow \mathbb{C}^{\Sigma_D},$$

by setting

$$j_J(\gamma, z) = (c_\mathrm{v} z_\mathrm{v}^J + d_\mathrm{v})_{\mathrm{v}\in\Sigma_D},$$

where $\gamma = \left(\left(\begin{smallmatrix} a_v & b_v \\ c_v & d_v \end{smallmatrix} \right) \right)_{v \in \Sigma_\infty}$ and

$$z_v^J = \begin{cases} z_v, & \text{if } v \in J, \\ \overline{z}_v, & \text{if } v \in \Sigma_D - J. \end{cases}$$

From the definition it is easy to verify that $j_J(\gamma\delta, z) = j_{J^\delta}(\gamma, \delta(z)) j_J(\delta, z)$.

Next we need to define the weights of our modular forms.

Definition 1.2.7. Let $n \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty}$ and $v \in \mathbb{Z}^{\Sigma_\infty}$ such that $n + 2v = (r, \ldots, r)$ for some $r \in \mathbb{Z}$. By abuse of notation we denote $(r, \ldots, r)$ by r for $r \in \mathbb{Z}$. Set $k = n + 2$ and $w = v + n + 1$. It follows from the above that all the entries of k have the same parity and $k = 2w - r$. We call the pair $(k, r) \in \mathbb{Z}_{\geq 2}^{\Sigma_\infty} \times \mathbb{Z}$ a ***classical (algebraic) weight.*** Note that given k (with all entries paritious and greater than 2) and r we can recover n, v, w. In what follows we will move between both descriptions when convenient. We will call (k, r, n, v, w) satisfying the above a ***weight tuple*** and usually denote it simply by (k, r).

Notation 1.2.8. If we take $k \geq 2$ paritious, then it is common to fix a choice of w, n, v, r as follows: let $k_0 = \max_i\{k_i\}$ then set $v = \left(\frac{k_0 - k_i}{2}\right)_i$, $n = k - 2$, $n_0 = k_0 - 2$, $r = n_0$ and $w = n + v + 1$. In this way if speak of a (classical) weight k Hilbert modular forms, where implicitly we mean we have k, w, n, v, r as above. Note that with this set-up we have $n + 2v = r$ and $w = \left(\frac{k_0 + k_i - 2}{2}\right)_i$.

We give a function $\mathbf{f} : G_D(\mathbb{A}) \to V(n^D, v^D, \mathbb{C})$ an action of $G_D(\mathbb{A})$ by setting

$$(\mathbf{f}|_{k,r,J}\gamma)(x) = j_{J^\gamma}(\gamma_\infty, \mathbf{i})^{-k_D} \operatorname{nrd}(\gamma_\infty)^{w_D} \mathbf{f}(x\gamma^{-1}) \cdot \gamma_\infty,$$

where $J \subset \Sigma_\infty$, $\gamma \in G_D(\mathbb{A})$ and $\mathbf{i} = (\sqrt{-1}, \ldots, \sqrt{-1})$. With this, the spaces of quaternionic modular forms are defined as follows:

Definition 1.2.9. For U an open compact subgroup of $G_D(\mathbb{A}_f)$ and $J \subset \Sigma_D$, we define $S_{k,r,J}^D(U)$ as the $\mathbb{C}$-vector space of functions

$$\mathbf{f} : G_D(\mathbb{A}) \longrightarrow V(n^D, v^D, \mathbb{C}),$$

such that:

(a) $\mathbf{f}|_{k,r,J}\gamma = \mathbf{f}$ for all $\gamma \in UC_\infty^+$.

(b) $\mathbf{f}(ax) = \mathbf{f}(x)$ for all $a \in G_D(\mathbb{Q})$.

(c) We now impose some holomorphy/antiholomorphy conditions. First, note that if we identify $\mathcal{H}$ with $\mathrm{GL}_2(\mathbb{R})/O_2(\mathbb{R})\mathbb{R}^\times$, then $G_{D,\infty}$ naturally acts on $\mathcal{H}^{\Sigma_D}$. Now, let $G_{D,\infty+}$ be the connected component of the identity in $G_{D,\infty}$. It is easy to see that for each $z \in \mathcal{H}^{\Sigma_D}$, we can choose $\gamma_\infty \in G_{D,\infty+}$ such that $\gamma_\infty(\mathbf{i}) = z$. From this we define

$$f_x : \mathcal{H}^{\Sigma_D} \longrightarrow V(n^D, v^D, \mathbb{C}),$$

for each $x \in G_D(\mathbb{A}_f)$ by setting

$$f_x(z) = j_J(\gamma_\infty, \mathbf{i})^{k_D} \operatorname{nrd}(\gamma_\infty)^{-w_D} \mathbf{f}(x\gamma_\infty) \cdot \gamma_\infty^{-1}.$$

Note $\mathbf{f}(ax) = \mathbf{f}(x)$ for all $a \in G_D(\mathbb{Q})$, which insures that this is well-defined, independent of the choice of γ_∞. With this we impose the condition that for all $x \in G_D(\mathbb{A}_f)$, $\frac{\partial f_x}{\partial \overline{z}_v} = 0$ if $v \in J$, and $\frac{\partial f_x}{\partial z_v} = 0$ if $v \in \Sigma_D - J$.

(d) When $D = M_2(F)$ we also require that

$$\int_{F\backslash\mathbb{A}_F} \mathbf{f}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0$$

for all $g \in G_D(\mathbb{A})$ and for each additive Haar measure dx on $F\backslash\mathbb{A}_F$. Furthermore, when $F = \mathbb{Q}$, we need $|\operatorname{Im}(z)^{k/2} f_x(z)|$ to be uniformly bounded on $\mathcal{H}$.

(e) When D is totally definite and the weight is $(2, r)$ we quotient out the space of forms that factor through the reduced norm map. In particular, if $S(U)$ is the space of functions satisfying (a) and (b) above, and $\operatorname{Inv}(U)$ is the subspace of $S(U)$ of functions that factor through nrd, then we let $S_{2,r}(U) = S(U)/\operatorname{Inv}(U)$. Note that in this case there are no J's since D is totally definite.

Remark 1.2.10. It is well-known that these spaces are finite dimensional. See (for example) [Gar90, Section 1.7].

Notation 1.2.11. (1) In the case when $\Sigma_D = \emptyset$ (the totally definite case) we drop the subscript J from $S^D_{k,r,J}(U)$.

(2) In the case $D = M_2(F)$ we drop the superscript D and denote the spaces simply as $S_{k,r,J}(U)$. Furthermore, in this case, if $J = \Sigma_\infty$ we again drop the subscript J.

1.3 Shimura varieties

Definition 1.3.1. Take U an open compact subgroup of $G_D(\mathbb{A}_f)$. We define the *Shimura variety*[3] associated to (G_D, U) to be

$$Y_D(U) = G_D(\mathbb{Q})\backslash G_D(\mathbb{A})/UC_\infty^+$$

(see 1.2.2 for the definition of C_∞^+).

Proposition 1.3.2. *For any given integral ideal $\mathfrak{n}$, one can find $t_i \in G_D(\mathbb{A})$, $i \in \{1, \dots h\}$ with the t_i having trivial infinite part and $(t_i)_\mathfrak{n} = 1$, such that*

$$G_D(\mathbb{A}) = \bigsqcup_{i=1}^{h} G_D(\mathbb{Q})t_i U G_{D,\infty+}.$$

Proof. This follows from strong approximation. $\square$

1.3.3. When D is indefinite, then $h = |F^\times\backslash\mathbb{A}_F^\times/\operatorname{nrd}(U)F_{\infty+}^\times|$ where $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes \widehat{\mathbb{Z}}$. In particular, for $M_2(F)$ and $\det(U) = \widehat{\mathcal{O}_F}^\times$, we have that h is the narrow class number of F. In this case we let $t_i \in \mathbb{A}_F^\times$ with $(t_i)_\infty = 1$ and such that $\mathfrak{t}_i = t_i\widehat{\mathcal{O}}_F \cap F$ is a complete set of representatives of the (narrow) ideal classes. Then setting $\left(\begin{smallmatrix} 1 & 0 \\ 0 & t_i \end{smallmatrix}\right)$ gives the required representatives. By abuse of notation we denote these representatives by t_i. For D totally definite, the number h depends on D and U and we call it the *class number of* (D, U).

Notation 1.3.4. For each i we set $\Gamma^i(U) = G_D(\mathbb{Q}) \cap t_i U t_i^{-1} G_{D,\infty+}$ (this is an intersection in $G_D(\mathbb{A})$) and $\overline{\Gamma}^i(U) = \Gamma^i(U)/\Gamma^i(U) \cap F^\times$.

For D indefinite define the complex analytic space $Y_{D,i}(U) = \Gamma^i(U)\backslash\mathcal{H}^{\Sigma_D}$; this is a manifold if $\overline{\Gamma}_i(U)$ has no torsion. Moreover,

$$Y_D(U) \cong \bigsqcup_i Y_{D,i}(U)$$

and this manifold will be compact if D is a division algebra; otherwise one needs to add cusps to get a compact space.

Proposition 1.3.5. *If D is totally definite then $Y_D(U)$ is a finite set of points.*

Proof. The work here is in proving that $G_D(\mathbb{Q})\backslash G_D(\mathbb{A})$ is compact, which follows from [Hid06, Theorem 2.8]. Once we have this, then since $Y_D(U)$ is the quotient of topological

[3] Some authors would call this a Shimura manifold as for some quaternion algebras this may not satisfy Deligne's axioms for a Shimura variety, but we will not follow this convention.

group by an open subgroup, it must be discrete. Therefore $Y_D(U)$ is compact and discrete, hence it is finite. $\square$

Definition 1.3.6. For D indefinite, we call an open compact subgroup U of $G_D(\mathbb{A}_f)$ *sufficiently small*, if for all i, $\overline{\Gamma}^i(U)$ has no torsion.

In practice we will be interested in the following subgroups.

Definition 1.3.7. Let $\mathfrak{n} = \prod_v \mathfrak{q}_v^{e_v}$ be an integral ideal. For D a quaternion algebra with $(\mathrm{Disc}(D), \mathfrak{n}) = 1$ we fix splitting at all primes dividing $\mathfrak{n}$ of D. Then we define:

$$\begin{aligned} U_1(\mathfrak{n}) &:= \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \mod \mathfrak{n} \right\}, \\ U_0(\mathfrak{n}) &:= \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod \mathfrak{n} \right\}, \\ U(\mathfrak{n}) &:= \left\{ \gamma \in (\mathcal{O}_D \otimes \widehat{\mathbb{Z}})^\times \mid \gamma \equiv 1 \mod \mathfrak{n} \right\}. \end{aligned}$$

Remark 1.3.8. By [Hid88, Lemma 7.1] one can always find $\mathfrak{n}$ such that the above are sufficiently small.

1.4 Hecke operators and the classical Jacquet-Langlands correspondence

In this section (following [Hid88]) we define the Hecke operators acting on the spaces of modular forms previously defined. With this we then state the classical Jacquet-Langlands correspondence, which will be one of the main results used later on.

Let U, U' be open compact subgroups of $G_D(\mathbb{A}_f)$ and $x \in G_D(\mathbb{A}_f)$. Now, write $UxU' = \coprod_i Ux_i$. Note that for $x = x_f x_\infty \in G_D(\mathbb{A}_f)C_\infty$ we have $Ux_fU' = \coprod_i U(x_i)_f$ if and only if $(UC_{\infty+})x(U'C_{\infty+}) = \coprod_i U(x_i)_f x_\infty$.

Definition 1.4.1. Let U, U' be as above and $\mathbf{f} \in S^D_{k,r,J}(U)$. Then define $[UxU'] : S^D_{k,r,J}(U) \to S^D_{k,r,J^x}(U')$ by

$$\mathbf{f}|[UxU'] = \sum_i \mathbf{f}|_{k,r,J} x_i.$$

The product of two such operators is defined by taking $U, U', U'' \in G_D(\mathbb{A}_f)$ open compact subgroups and $x, y \in G_D(\mathbb{A}_f)$ and then noting that $UxU'yU'' = \coprod_i UxU'y_i = \coprod_{i,j} Ux_jy_i$. From this, one obtains operators

$$[UxU'][U'yU''] : S^D_{k,r,J}(U) \longrightarrow S^D_{k,r,J^{xy}}(U'').$$

Definition 1.4.2. Let U be as above and let Δ be a subsemigroup of $G_D(\mathbb{A}_f)$ such that $U \subset \Delta$. Then the $\mathbb{Z}$-module of all finite formal sums of $[UxU]$ for $x \in \Delta$ is an associative ring under the product defined above. We call this the *Hecke ring associated to* (U, Δ) and we denote it by $\boldsymbol{T}^D(U, \Delta)$.

In practice, we will take Δ to be the following:

Definition 1.4.3. Let $\mathfrak{n}$ be an ideal coprime to $\mathrm{Disc}(D)$ and fix splittings at all places away from $\mathrm{Disc}(D)$. Let

$$\Delta_D(\mathfrak{n}) = \left\{\gamma \in G_D(\mathbb{A}_f) \mid \gamma_{\mathrm{v}} = \left(\begin{smallmatrix} a_{\mathrm{v}} & b_{\mathrm{v}} \\ c_{\mathrm{v}} & d_{\mathrm{v}} \end{smallmatrix}\right) \in M_2(\mathcal{O}_{\mathrm{v}}) \text{ with } d_{\mathrm{v}} \in \mathcal{O}_{\mathrm{v}}^{\times}, c_{\mathrm{v}} \in \mathfrak{n}_{\mathrm{v}} \text{ for all } \mathrm{v}|\mathfrak{n}\right\}.$$

For $U = U_*(\mathfrak{n})$, we let $\boldsymbol{T}^D(U) = \boldsymbol{T}^D(U, \Delta_D(\mathfrak{n}))$.

One particularly useful operator (as shown by the proposition below) is given by $[Ux_JU]$ where $J \in \Sigma_D$ and x_J is as in 1.2.5.

Proposition 1.4.4. *For each $J \in \Sigma_D$ the map $[Ux_JU] : S_{k,r,J}(U) \longrightarrow S_{k,r,\Sigma_D}(U)$ is an isomorphism which commutes with $[UyU]$ for all $y \in G_D(\mathbb{A}_f)$.*

Proof. First we note that since $x_{J,f} = 1$ we must have that $[Ux_JU] \circ [UyU] = [UyU] \circ [Ux_JU]$. Moreover, note that $\Sigma_D^{x_J} = J$, therefore $[Ux_JU]^2 = 1$ and thus $[Ux_JU]$ is an isomorphism. □

This then shows that the Hecke action on $S^D_{k,r,J}(U)$ is independent of J, so if we only care about the Hecke action, there is no loss in dropping the subscript J. Now one might ask to what extent is the Hecke action independent of D. For this there is the following very important theorem.

Theorem 1.4.5. *(Eichler, Jacquet–Langlands, Shimizu). Let D be a division quaternion algebra over a totally real field F, with $\mathrm{Disc}(D) = \mathfrak{d}$ and $\mathfrak{n}$ a ideal coprime to $\mathfrak{d}$. Then we have an isomorphism of $\boldsymbol{T}^D(U_*(\mathfrak{n}))$ Hecke modules*

$$S^D_{k,r}(U_*(\mathfrak{n})) \cong S^{\mathfrak{d}\text{-}new}_{k,r}(U_*(\mathfrak{n}\mathfrak{d}))$$

where $* \in \{0, 1, \emptyset\}$.

Proof. This result is just a concrete realization of [JL70, Theorem 16.1]. □

Note that we have not yet defined $S^{\mathfrak{d}\text{-new}}_{k,r}(U_i(\mathfrak{n}\mathfrak{d}))$ but this will be done in Definition 3.2.10. Also in the above we have denoted the level structures for D and $M_2(F)$ by $U_i(-)$ which is a slight abuse of notation. We have also identified the Hecke operators away from $\mathfrak{d}$ for D and G via the fixed splittings of D.

Chapter 2

Eigenvarieties

In this chapter we will give some background on eigenvarieties and how to induce maps between them. We will begin by giving some properties of Banach modules and compact operators on them, from which we will later construct eigenvarieties. There are many sources for this material and we shall mainly follow [Buz07, Han, Urb11].

2.1 Banach modules

Definition 2.1.1. Let L be a complete non-archimedean field with norm $|.|_L$. A (non-zero) commutative Noetherian L-*Banach algebra* A is a commutative Noetherian L-algebra[1] complete with respect to the metric induced by a norm $|.| : A \to \mathbb{R}_{\geq 0}$ satisfying

(a) For $a, b \in A$, $|ab| \leq |a| \cdot |b|$.

(b) For $a \in A$ and $\lambda \in L$, $|\lambda a| = |\lambda|_L |a|$.

Definition 2.1.2. Let A be a commutative Noetherian L-Banach algebra. An A-module M is called a *Banach A-module* if it is complete with respect to $|.| : M \to \mathbb{R}_{\geq 0}$ satisfying

(a) For $m \in M$, $|m| = 0$ if and only if $m = 0$.

(b) For $m, n \in M$, $|m + n| \leq \max\{|m|, |n|\}$.

(c) For $a \in A$ and $m \in M$, $|am| \leq |a||m|$.

It follows from [Buz07, Proposition 2.1 (b)] that finite Banach A-module has a canonical topology induced by any norm making it into a Banach A-module. We will always assume that our modules have this topology.

[1]Throughout, our algebras will always be unital.

Notation 2.1.3. For I an indexing set and $a_i \in A$ for $i \in I$, we say that $\lim_i a_i = 0$ if for all $\epsilon > 0$ there are only finitely many $i \in I$ with $|a_i| > \epsilon$.

Definition 2.1.4. Let A, M be as above and I an indexing set. A subset $\{e_i \in M \mid i \in I\}$ with $|e_i| = 1$ for all $i \in I$ as called an ***orthonormal basis for*** M if the following holds:

(a) Every $m \in M$ can be uniquely written as $\sum_i a_i e_i$ with $a_i \in A$ and $\lim_i a_i = 0$.

(b) If $m = \sum_i a_i e_i$, then $|m| = \max_i\{|a_i|\}$.

Such a module M is called ***orthonormalizable*** or simply ***ON-able***. More generally, M is called ***potentially ON-able*** if there exists a norm on M equivalent to the given norm under which M becomes ON-able.

Definition 2.1.5. Let M, N be ON-able Banach A modules with bases $\{e_i | i \in I\}$ and $\{f_j | j \in J\}$ respectively. Then, for $\phi : M \to N$ a continuous A-module homomorphism, we define the ***associated matrix coefficients*** $(a_{i,j})$ by $\phi(e_i) = \sum_j a_{j,i} f_j$. [2]

Definition 2.1.6. We say a Banach A-module P satisfies property (Pr) if there is a Banach A-module Q, such that $P \oplus Q$ (with its usual norm) is potentially ON-able.

2.2 Compact operators and slope decompositions

We now collect some results on compact operators on Banach modules and the induced slope decompositions, which is crucial for the construction of eigenvarieties.

Definition 2.2.1. Let $L/\mathbb{Q}_p$ be a finite extension and $Q(X) \in L[X]$ be a polynomial of degree d. We say Q has ***slope-***$\leq h$, if $Q(0) \in \mathcal{O}_L^\times$ and if the roots of $Q^*(X) := x^d Q(1/X)$ in $\overline{\mathbb{Q}}_p$ have p-adic valuation less than or equal to h.

Definition 2.2.2. Let M be a vector space over L and U a (continuous) linear endomorphism of M. We say that M has a ***slope-***$\leq h$ ***decomposition with respect to*** U, if we can write $M := M_1 \oplus M_2$, where both M_1, M_2 are stable under the action of U and

(a) M_1 is finite dimensional over L;

(b) the polynomial $\det(1 - XU|_{M_1})$ is of slope-$\leq h$;

(c) for any polynomial P of slope-$\leq h$, the restriction of $P^*(U)$ to M_2 is an automorphism of M_2.

[2]This is *contrary* to Serre's convention in [Ser62].

Lemma 2.2.3. *Let M, N be L-vector spaces, and let U, V be endomorphisms of M, N respectively. If $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ are slope-$\leq h$ decompositions with respect to U, V respectively, and $f : M \to N$ is a (continuous) L-linear map such that $f \circ U = V \circ f$, then $f(M_i) \subset N_i$ for $i = 1, 2$.*

Proof. See [Urb11, Lemma 2.3.2]. □

Setting $M = N$ and $f = id$ gives us uniqueness of the slope-$\leq h$ decomposition, which means that there is no problem in defining $M^{\leq h} := M_1$ and $M^{>h} := M_2$ for M_1, M_2 as above. Moreover, note that for $h' \geq h$, if M has a slope-$\leq h'$ decomposition, then it also has a slope-$\leq h$ decomposition and there is a U stable decomposition $M^{\leq h'} = M^{\leq h} \oplus M^{>h, \leq h'}$.

Remark 2.2.4. It is easy to see that if M has a slope-$\leq h$ decomposition with respect to U, then for $\alpha \in L$, $V = \alpha U$ gives a slope-$\leq h + \mathrm{val}_p(\alpha)$ decomposition and

$$M(U)^{\leq h} = M(V)^{\leq h + \mathrm{val}_p(\alpha)},$$

where $M(X)$ denotes the slope decomposition of M with respect to X.

Definition 2.2.5. We say that M has a slope decomposition with respect to U, if there is a sequence of rationals h_n going to infinity (hence for any such sequence), such that M has a slope-$\leq h_n$ decomposition for all $n \in \mathbb{N}$.

Notation 2.2.6. We set $\det(1 - XU\mid_M) := \lim_n \det(1 - XU|_{M^{\leq h_n}})$. Note that the limit exists in $L[\![X]\!]$ and it does not depend on the sequence (h_n).

Proposition 2.2.7. *Let M have a slope decomposition with respect to U and $M' \subset M$ be a U-stable subspace of M. Then M' has a slope decomposition if and only if M/M' also has a slope decomposition and furthermore*

$$\det(1 - XU\mid_M) = \det(1 - XU\mid_{M'}) \det(1 - XU\mid_{M/M'}).$$

Proof. See [Urb11, Corollary 2.3.5]. □

Now we switch from vector spaces M to ON-able Banach A-modules, where A is a topologically finitely generated $\mathbb{Q}_p$-Banach algebra. But note (as in [Urb11, Section 2]) that the theory we are about to develop works just as well when M is a compact p-adic Fréchet space, which we recall is a p-adic topological vector space V which is the limit of p-adic Banach space V_n, such that the transition maps $V_n \to V_m$ for $n > m$ are compact (defined below).

Definition 2.2.8. Let M be an ON-able Banach A-algebra. We call an operator U *compact* (or *completely continuous*) if there exists a sequence of projective and finitely generated Banach A-modules M_i, such that $U_i := U|_{M_i}$ converges (with respect to the operator norm) to U as $i \to \infty$.

In the case when our module M is ON-able, then we can use the matrix associated to the operator to 'see' when an operator is compact as follows.

Proposition 2.2.9. *Let M, N be ON-able Banach A-modules with ON-bases $\{e_i | i \in I\}$ and $\{f_j | j \in J\}$ respectively. Let $\phi : M \to N$ be a continuous A-module homomorphism with matrix $(a_{i,j})$. Then ϕ is compact if and only if $\lim_{i\to\infty} \sup_{j \in I} |a_{i,j}| = 0$.*

Proof. This is [Buz07, Proposition 2.4]. □

Definition 2.2.10. Let M, M_i and U_i be as in Definition 2.2.8. We define the Fredholm determinant as

$$\mathrm{Fred}_M(U) = \det(1 - XU|_M) := \lim_i \det(1 - XU_i).$$

Definition 2.2.11. If A is a local ring with maximal ideal $\mathfrak{m}$ and $F(X) \in A[\![X]\!]$. Then F is called *entire over* A if the n-th coefficients of F lies in $\mathfrak{m}^{c_n}$ for some $c_n \in \mathbb{Z}$ such that c_n/n tends to ∞.

Theorem 2.2.12. *Let A, M and U be as above, then $\mathrm{Fred}_M(U)$ is an entire power series with coefficients in A.*

Proof. This follows from [Ser62, Proposition 7]. □

Definition 2.2.13. If we can write $\mathrm{Fred}_M(U) = Q(X) \cdot R(X)$, where Q is a multiplicative polynomial of slope-$\leq h$ and $R(X) \in A[\![X]\!]$ is an entire power series of slope $> h$ (meaning its Newton polygon has all of its slopes greater that h), then we say that $\mathrm{Fred}_M(U)$ has a *slope-$\leq h$ factorization.*

Proposition 2.2.14. *With the notation as above, M has a slope-$\leq h$ decomposition if and only if $\mathrm{Fred}_M(U)$ admits a slope-$\leq h$ factorization.*

Proof. See [Buz07, Theorem 3.3]. □

Note that the slope-$\leq h$ factorization of $\mathrm{Fred}_M(U) = Q(X)R(X)$ gives the slope-$\leq h$ decomposition on M by setting $M^{\leq h} = \{m \in M \mid Q^*(U) \cdot m = 0\}$.

Definition 2.2.15. For M as above with a slope decomposition with respect to U, let

$$M_{\text{fs}} = \bigcup_{h<\infty} M^{\leq h},$$

which we call the *finite slope* part of M.

2.3 The spectral variety

We begin by recalling the following definitions from rigid geometry.

Definition 2.3.1. If R is an affinoid integral domain, we say that R is *relatively factorial* if for any $f \in R\langle X\rangle$ (which is the space of convergent power series with coefficients in R) with constant term 1, the ideal (f) factors uniquely as a product of principal prime ideals, where each prime ideal can be chosen to be generated by an element of constant term 1. Furthermore, we say that a rigid analytic space is *relatively factorial* if it has an admissible covering by relatively factorial affinoids.

Definition 2.3.2. If $\mathcal{W}$ is a relatively factorial rigid analytic space and $i : \mathcal{W} \times \{0\} \hookrightarrow \mathcal{W} \times \mathbb{A}^1$ is the natural map, inducing a map $i^* : \mathscr{O}(\mathcal{W} \times \mathbb{A}^1) \to \mathscr{O}(\mathcal{W})$, then a *Fredholm series* is a global section $f \in \mathscr{O}(\mathcal{W} \times \mathbb{A}^1)$ such that $i^*(f) = 1$. The subspace of $\mathcal{W} \times \mathbb{A}^1$ cut out by a Fredholm series is called a *Fredholm hypersurface.*

Proposition 2.3.3. *If f is a Fredholm series and $\mathscr{Z}(f)$ is the Fredholm hypersurface it defines, then the natural map $\mathcal{W} \times \mathbb{A}^1 \to \mathcal{W}$ induces a map $\mathscr{Z} \to \mathcal{W}$ whose image is Zariski open in $\mathcal{W}$.*

Proof. See (for example) [Han, Proposition 4.1.3]. □

Definition 2.3.4. Let $\mathfrak{U} \subset \mathcal{W}$ be an affinoid and $\mathscr{Z}(f)$ a Fredholm hypersurface. Define

$$\mathscr{Z}_{\mathfrak{U},h} = \mathscr{O}(\mathfrak{U})\langle p^h X\rangle/(f(X)),$$

which we view as an admissible affinoid open subset of $\mathscr{Z}(f)$. We have a natural map $\mathscr{Z}_{\mathfrak{U},h} \to \mathfrak{U}$, which is flat but might not be finite. We say that $\mathscr{Z}_{\mathfrak{U},h}$ is *slope-adapted* if the above map is finite and flat.

It is possible to show that $\mathscr{Z}_{\mathfrak{U},h}$ is slope-adapted if and only if $f|_{\mathfrak{U}}$ admits a slope-$\leq h$ factorization $f|_{\mathfrak{U}}(X) = Q(X)R(X)$, from which it follows that $\mathscr{O}(\mathscr{Z}_{\mathfrak{U},h}) = \mathscr{O}(\mathfrak{U})[X]/(Q(X))$. Then, [Buz07, Theorem 4.6] tells us that the collection slope-adapted affinoids is an admissible cover of $\mathscr{Z}(f)$. It is this fact that allows us to construct eigenvarieties.

2.4 Eigenvarieties

In order to define an eigenvariety $\mathscr{X}$, we need to specify the eigendata to which it is associated. The constructions is originally due to Coleman-Mazur and Buzzard, but has been extended by Urban and Hansen among others.

We begin by specifying the eigendata as defined [Buz07] to which one can attach an equidimensional eigenvariety, by using Buzzard's eigenmachine. Later we will use this eigenmachine to construct eigenvarieties attached to D/F totally definite or GL_2/F as in [Buz07, AIP16b].

More generally, Hansen has given a more general construction of eigenvarieties, which can be use to construct eigenvarieties associated to reductive groups by using overconvergent cohomology. These more general eigenvarieties may not be equidimensional. We will recall this more general construction and later use this to construct eigenvarieties attached to any quaternion algebra D/F.

2.4.1 Buzzard's Eigenmachine

We begin by recalling some definitions from [Buz07].

Definition 2.4.2. Let M_1, M_2 be Banach R-modules satisfying (Pr) for R a reduced affinoid and $\boldsymbol{T}$ a commutative R-algebra with maps $\psi_i : \boldsymbol{T} \to \mathrm{End}_R(M_i)$. Let $U \in \boldsymbol{T}$ act compactly on both M_1 and M_2. A continuous R-module and $\boldsymbol{T}$-module homomorphism $\alpha : M_1 \to M_2$ is called a ***primitive link*** if there is a compact R-linear and $\boldsymbol{T}$-linear map $c : M_2 \to M_1$ such that $\psi_2(U) : M_2 \to M_2$ is $\alpha \circ c$ and $\psi_1(U) : M_1 \to M_1$ is $c \circ \alpha$. More generally a continuous R-module and $\boldsymbol{T}$-module homomorphism $\alpha : M' \to M$ is a ***link*** if there exists a sequence M_i of Banach R-modules satisfying (Pr) for $i \in \{0, \ldots, n\}$ such that $M' = M_0$, $M = M_n$ and α factors as a compositum of maps $\alpha_i : M_i \to M_{i+1}$ with α_i a primitive link.

Definition 2.4.3. Let $\mathcal{W}$ be a reduced rigid space, R a reduced affinoid and $\boldsymbol{T}$ be a commutative R-algebra with a specified element U. For admissible affinoid open $\mathfrak{U} \subset \mathcal{W}$ let $M(\mathfrak{U})$ a Banach $\mathscr{O}(\mathfrak{U})$-module satisfying (Pr) with an R-module homomorphism $\psi_{\mathfrak{U}} : \boldsymbol{T} \to \mathrm{End}_{\mathscr{O}(\mathfrak{U})}(M_{\mathfrak{U}})$ such that $\psi_{\mathfrak{U}}(U)$ is compact. Finally assume that if $\mathfrak{U} \subset \mathfrak{U}' \in \mathcal{W}$ are two admissible affinoid opens, then there is a continuous $\mathscr{O}(\mathfrak{U})$-module homomorphism $\alpha : M_{\mathfrak{U}} \to M_{\mathfrak{U}'} \hat{\otimes}_{\mathscr{O}(\mathfrak{U}')} \mathscr{O}(\mathfrak{U})$ which is a link and such that if $\mathfrak{U}_1 \subset \mathfrak{U}_2 \subset \mathfrak{U}_2 \subset \mathcal{W}$ are all affinoid subdomains then $\alpha_{13} = \alpha_{23} \circ \alpha_{12}$ for $\alpha_{ij} : M_{\mathfrak{U}_i} \to M_{\mathfrak{U}_i} \widehat{\otimes}_{\mathscr{O}(\mathfrak{U}_i)} \mathscr{O}(\mathfrak{U}_j)$.

We give the name of ***eigendata*** or ***eigenvariety data***, to tuple $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$ where $\mathscr{M}$ is the coherent sheaf defined by the $M_{\mathfrak{U}}$.

Definition 2.4.4. For U as above, define the *spectral variety associated to* U, denoted $\mathscr{Z}(U)$ as the closed subspace of $\mathcal{W} \times \mathbb{A}^1$ cut out be the Fredholm determinant of U.

With this we have the following theorem of Buzzard:

Theorem 2.4.5 (The Eigenmachine). *Attached to $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$ there is a canonically associated rigid space $\mathscr{X}(\mathfrak{E})$ with a finite morphism to the spectral variety $\mathscr{Z}(U)$ defined by U and whose points over $z \in \mathscr{Z}(U)$ are in bijection with the generalized eigenspace for the action of $\boldsymbol{T}$ on the fibre $\mathscr{M}_z$. Moreover, if $\mathcal{W}$ is equidimensional of dimension n, then so is $\mathscr{X}(\mathfrak{E})$.*

Proof. This follows from [Buz07, Construction 5.7, Lemmas 5.8-5.9]. □

2.4.6 Hansen's Eigenmachine

We now recall Hansen's more general construction of eigenvarieties. For these we do not need to have a sheaf of Banach modules satisfying (Pr) as in Buzzard's construction, but the resulting eigenvarieties may not be equidimensional.

Definition 2.4.7. We give the name of *generalized eigendata* or *generalized eigenvariety data* to the tuple

$$\mathfrak{D} = (\mathcal{W}, \mathscr{Z}, \mathscr{M}, \boldsymbol{T}, \psi),$$

where:

1. $\mathcal{W}$ is the 'weight space', which is a separated, reduced, equidimensional, relatively factorial rigid analytic space,
2. $\mathscr{Z} \subset \mathcal{W} \times \mathbb{A}^1$ is a Fredholm hypersurface,
3. $\mathscr{M}$ is a coherent analytic sheaf on $\mathscr{Z}$,
4. $\boldsymbol{T}$ a commutative $\mathbb{Q}_p$-algebra (the Hecke algebra),
5. ψ is a $\mathbb{Q}_p$-algebra homomorphism $\psi : \boldsymbol{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{M})$.

Theorem 2.4.8 (The Generalized Eigenmachine). *Attached to the eigendata $\mathfrak{D} = (\mathcal{W}, \mathscr{Z}, \mathscr{M}, \boldsymbol{T}, \psi)$, there is a rigid analytic space $\mathscr{X} := \mathscr{X}(\mathfrak{D})$, together with a finite morphism $\pi : \mathscr{X} \to \mathscr{Z}$, an algebra homomorphism*

$$\phi_{\mathscr{X}} : \boldsymbol{T} \longrightarrow \mathscr{O}(\mathscr{X}),$$

and a coherent sheaf $\mathscr{M}'$ on $\mathscr{X}$ with a canonical isomorphism $\pi_*(\mathscr{M}') \cong \mathscr{M}$ compatible with the action of $\boldsymbol{T}$. The points of $\mathscr{X}$ lying over $z \in \mathscr{Z}$ are in bijection with the generalized eigenspaces for the action of $\boldsymbol{T}$ on $\mathscr{M}_z$.

Proof. This is [Han, Theorem 4.2.2].

□

2.4.9. We will construct eigenvarieties in this setting by taking some weight space $\mathcal{W}$ and to each affinoid open $\mathfrak{U} \subset \mathcal{W}$ attaches a nuclear Frechet space $H^*(Y_D(U), \mathscr{D}_{\mathfrak{U}})$. On this space we will have a compact operator U_p whose Fredholm determinant $F_{\mathfrak{U}}$ which one checks (using links) is well-defined independent of $s \geq s(\mathfrak{U})$ (cf. [Han, Section 3.1]) and such that for $\mathfrak{U}' \subset \mathfrak{U}$ open we have $F_{\mathfrak{U}} |_{\mathfrak{U}'}= F_{\mathfrak{U}'}$, from which it follows (by Tate's acyclity theorem) that there is a 'global' F such that $F_{\mathfrak{U}} = F |_{\mathfrak{U}}$. The spectral variety $\mathscr{Z}$ is then given by the zero locus of F in $\mathcal{W} \times \mathbb{A}^1$.

In order to construct the coherent sheaf $\mathscr{M}$ on $\mathscr{Z}$ one simply defines $\mathscr{M}(\mathscr{Z}_{\mathfrak{U},h}) = H^*(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h}$ for $\mathscr{Z}_{\mathfrak{U},h}$ a slope-adapted affinoid, which one can then glue to get a coherent sheaf on $\mathscr{Z}$ (cf. [Han, Section 4.3]).

Remark 2.4.10. The main difference between eigenvarieties constructed via the eigenmachine compared to the generalized eigenmachine is the support of the relevant system of Banach or Frechet modules. In Buzzard's eigenmachine they are Banach modules satisfying (Pr) and are supported on all of $\mathscr{Z}(U)$ while in Hansen's construction one allows more general nuclear Frechet spaces which are supported on a subspace of $\mathscr{Z}$ of possibly positive codimension (cf. [Han, Section 1.1 and Section 4.4]).

Remark 2.4.11. One can obtain a set of generalized eigendata from Buzzard's eigendata $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U_p)$ by defining $\mathfrak{D} = (\mathcal{W}, \mathscr{Z}(U), \mathscr{M}, \boldsymbol{T}, \psi)$ where $\mathscr{Z}(U)$ is as in Definition 2.4.4 and $\psi : \boldsymbol{T} \to \mathrm{End}_{\mathcal{O}_{\mathscr{Z}(U)}}(\mathscr{M})$ the algebra homomorphism naturally defined by the $\psi_{\mathfrak{U}}$ as in Definition 2.4.3. The resulting eigenvariety coincides with the one given by Buzzard's eigenmachine.

2.5 The Interpolation Theorem

In this section we begin by recalling Chenevier's interpolation theorem, which is used to construct closed immersions between eigenvarieties (as constructed by Buzzard's eigenmachine) which interpolate 'classical maps', a process which is referred to as p-adic Langlands functoriality. We will use this to interpolate the Jacquet-Langlands correspondence. For this we will need to find a very Zariski dense subset (see Definition 2.5.2) of the weight space, together with a classical structure on it. As the name suggests, the classical structure will be given by the subspace of classical modular forms inside the space of overconvergent modular forms. We then use this to find closed immersions between different eigenvarieties by relating their classical structures. In our case, it will be the classical Jacquet-Langlands correspondence that will allow us to relate the classical structures.

In order to construct maps between eigenvarieties using the generalized eigenmachine we will need a more general version of this interpolation theorem due to Hansen, which we will use in Section 6.2.

Definition 2.5.1. A subset $X \subset Z$ is *Zariski dense* in Z if for every analytic subset (see [BGR84, Section 9.5.2]) $Y \subset Z$ such that $X \subset Y$, then $Y = Z$.

Definition 2.5.2. A Zariski dense subset $X \subset \mathcal{W}$ is *very Zariski dense* if for each $x \in X$ and every affinoid open $V \subset \mathcal{W}$ containing x, $V \cap X$ is Zariski dense in each irreducible component of V containing x.

From this we define the classical structures as follows.

Definition 2.5.3. Let $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$ be a set of eigendata as above and let $X \subset \mathcal{W}$ be a very Zariski dense subset. For each $x \in X$, let $\mathscr{M}_x^{cl}$ be a finite dimensional $\boldsymbol{T}$-module contained in $\mathscr{M}_x$ and, for every $h \in \mathbb{R}$, set $X_h = \{x \in X \mid \mathscr{M}_x^{\leq h} \subset \mathscr{M}_x^{cl}\}$. We say that $\mathscr{M}^{cl}$ gives a *classical structure* on X if for every open affinoid neighbourhood $V \subset \mathcal{W}$ and every h, the sets $X \cap V, X_h \cap V$ have the same Zariski closure in V.

Definition 2.5.4. If $\mathscr{X}$ is an eigenvariety, with eigendata $\mathfrak{E} = (\mathcal{W}, \mathscr{M}, \boldsymbol{T}, U)$, we denote the *nilreduction* of $\mathscr{X}$ by $\mathscr{X}^{red}$, and we say that an eigenvariety is *reduced* if $\mathscr{X}^{red} \cong \mathscr{X}$.

With these definitions we can now state the Chenevier's *Interpolation theorem.*

Theorem 2.5.5. *(Chenevier) Let $\mathscr{X}_i$ be eigenvarieties associated to the eigendata of $\mathfrak{E}_i = (\mathcal{W}_i, \mathscr{M}_i, \boldsymbol{T}_i, \psi_i)$, for $i = 1, 2$ with $\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2$ and $\boldsymbol{T} = \boldsymbol{T}_1 = \boldsymbol{T}_2$. Let $X \subset \mathcal{W}$ a very Zariski dense subset such that $\mathscr{M}_i^{cl}$ is a classical structure on X for each $\mathscr{M}_i$. Assume that, for all $t \in \boldsymbol{T}$ and all $x \in X$, we have*

$$\det\left(1 - \psi_1(tU)Y \mid_{\mathscr{M}_{1,x}^{cl}}\right) \ \textit{divides} \ \det\left(1 - \psi_2(tU)Y \mid_{\mathscr{M}_{2,x}^{cl}}\right)$$

in $k(x)[Y]$, where $k(x)$ is the residue field at x. Then, there is a canonical closed immersion $\iota : \mathscr{X}_1^{red} \hookrightarrow \mathscr{X}_2^{red}$ such that the following diagrams commute

$$\begin{tikzcd} \mathscr{X}_1^{red} \arrow[r, hook, "\iota"] \arrow[d] & \mathscr{X}_2^{red} \arrow[dl] \\ \mathcal{W} & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{T} \arrow[r, "\phi_1^{red}"] \arrow[d, "\phi_2^{red}"'] & \mathscr{O}(\mathscr{X}_1^{red}) \\ \mathscr{O}(\mathscr{X}_2^{red}) \arrow[ur, "\iota^*"] & \end{tikzcd}$$

Proof. See [Che05, Theorem 1]. $\square$

Corollary 2.5.6. *If* $\det\left(1-\psi_1(tU)Y_{|\mathscr{M}^{cl}_{1,x}}\right)=\det\left(1-\psi_2(tU)Y_{|\mathscr{M}^{cl}_{2,x}}\right)$ *in* $k(x)[Y]$ *for all* $t\in\boldsymbol{T}$ *and all* $x\in X$*, then there is an isomorphism* $\mathscr{X}_1^{red}\cong\mathscr{X}_2^{red}$*.*

Proof. In this case the above Theorem gives us two closed immersions $\iota_{12}:\mathscr{X}_1^{red}\hookrightarrow\mathscr{X}_2^{red}$ and $\iota_{21}:\mathscr{X}_2^{red}\hookrightarrow\mathscr{X}_1^{red}$, from which the result follows at once by noting that $\iota_{12}\iota_{21}=Id_{\mathscr{X}_2}$ and $\iota_{21}\iota_{12}=Id_{\mathscr{X}_1}$. $\square$

We now have a result of Chenevier that gives a criterion for an eigenvariety to be reduced. Suppose that $X\subset\mathcal{W}$ is a very Zariski dense subset giving a classical structure. For $h\in\mathbb{R}$, let $X_h^{ss}=\{x\in X\mid\mathscr{M}_x^{cl}\cap\mathscr{M}_x^{\leq h}$ is a semisimple $\boldsymbol{T}$-module$\}$.

Lemma 2.5.7. *If for all* $h\in\mathbb{R}$*,* $x\in X$ *and* $V\subset\mathcal{W}$ *an open affinoid containing* x*, there exists* $W\subset V$ *an open affinoid containing* x*, such that* $X_h^{ss}\cap W$ *contains an open Zariski dense subset of* $X\cap W$*, then* $\mathscr{X}$ *is reduced (here we view* $X\cap W$ *as a topological subspace of* W *with the Zariski topology).*

Proof. See [Che05, Proposition 3.9]. $\square$

Now for more general eigenvarieties as constructed in using Hansen's eigenmachine we have a more general interpolation theorem. Before stating it, we need to recall some definitions from [Han].

Definition 2.5.8. Let $\mathscr{X}=\mathscr{X}(\mathfrak{D})$ be an eigenvariety attached to a generalized eigendatum of $\mathfrak{D}=(\mathcal{W},\mathscr{Z},\mathscr{M},\boldsymbol{T},\psi)$. The *core* $\mathscr{X}^o$ of $\mathscr{X}$ is defined the union of the $\dim(\mathcal{W})$-dimensional irreducible components of $\mathscr{X}^{red}$ regarded as a closed subspace of $\mathscr{X}$ and let $\mathscr{Z}^o$ be the subspace of $\mathscr{Z}$ of points whose preimage in $\mathscr{X}$ meets $\mathscr{X}^o$. An eigenvariety is called *unmixed* if $\mathscr{X}^o\cong\mathscr{X}$.

Remark 2.5.9. Note that eigenvarieties constructed using Buzzard's eigenmachine will be unmixed if they are reduced.

Theorem 2.5.10 (Hansen)**.** *Let* $\mathscr{X}_i$ *be eigenvarieties associated to the generalized eigendata* $\mathfrak{D}_i=(\mathcal{W}_i,\mathscr{Z}_i,\mathscr{M}_i,\boldsymbol{T}_i,\psi_i)$ *for* $i=1,2$ *with* $\boldsymbol{T}=\boldsymbol{T}_1=\boldsymbol{T}_2$ *together with:*

1. *A closed immersion* $j:\mathcal{W}_1\hookrightarrow\mathcal{W}_2$*.*

2. *A very Zariski dense subset $\mathscr{Z}^{cl} \subset \mathscr{Z}_1^o$ with image in $\mathscr{Z}_2$ under the map induced by j and such that for all $t \in \boldsymbol{T}$ and all $z \in \mathscr{Z}^{cl}$*

$$\det\left(1 - \psi_1((tU))Y \mid_{\mathscr{M}^{cl}_{1,z}}\right) \textit{ divides } \det\left(1 - \psi_2(tU)Y \mid_{\mathscr{M}^{cl}_{2,z}}\right)$$

in $k(z)[Y]$.

Then j induces a closed immersion $\mathscr{Z}_1^o \hookrightarrow \mathscr{Z}_2$ and there is a canonical closed immersion $\iota : \mathscr{X}_1^o \hookrightarrow \mathscr{X}_2$ such that the following diagrams commute

$$\begin{tikzcd} \mathscr{X}_1^o \arrow[r, hook, "\iota"] \arrow[d] & \mathscr{X}_2 \arrow[d] \\ \mathcal{W}_1 \arrow[r, hook, "j"] & \mathcal{W}_2 \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{T} \arrow[r, "\phi_1^o"] \arrow[d, "\phi_2"] & \mathscr{O}(\mathscr{X}_1^o) \\ \mathscr{O}(\mathscr{X}_2^{red}) \arrow[ur, "\iota^*"] & \end{tikzcd}$$

Proof. This is [Han, Theorem 5.1.2]. □

Remark 2.5.11. Recently, Johansson-Newton [JN17, Section 3.1] have given a more general version of this interpolation theorem for "adic" eigenvarieties.

Part II

The overconvergent Jacquet-Langlands correspondence

In this section we begin by studying classical Hilbert modular forms in detail and understanding the action of Hecke operators by using q-expansions. We then give a more geometric definition (a la Katz) of these spaces, which naturally generalizes to the p-adic setting and is the basis for defining overconvergent Hilbert modular forms as defined in [AIP16b]. From this one constructs eigenvarieties associated to Hilbert modular form, which we will denote by $\mathscr{X}_G(U)$, for U an appropriate level.

We then define quaternionic modular forms in the p-adic setting for a totally definite quaternion algebra and define overconvergent quaternionic modular forms following [Buz07]. We also prove a control theorem in this setting and construct eigenvarieties $\mathscr{X}_D(U)$.

Similarly, we define eigenvarieties associated to overconvergent cohomology groups on any quaternion algebra D. In this case one does not need to worry about moduli problems being representable (in contrast to the Hilbert modular form case).

Lastly, using the Interpolation theorem we show that, if $|F : \mathbb{Q}|$ is even and $\mathrm{Disc}(D) = 1$, then $\mathscr{X}_D(U) \cong \mathscr{X}_G(U)$. More generally, using overconvergent cohomology, we obtain closed immersions between eigenvarieties associated to any quaternion algebra D by using Hansen's interpolation theorem.

Chapter 3

Hilbert modular forms

In this chapter we begin by recording some classical results about Hilbert modular forms. We will start by making explicit the definition of Hilbert modular forms by setting $D = M_2(F)$ in Definition 1.2.9. We will then discuss their q-expansions and study the action of Hecke operators on these spaces; in particular, we record some 'Atkin-Lehner type' results. We will also give a more geometric constructions of these spaces, which naturally generalize to the p-adic setting. Much of this material is well-known and there are many sources, see [Hid88, Shi78a, SW93, Dim03, Gar90].

We will then recall the definition of overconvergent Hilbert modular forms given by [AIP16b] which are used to construct eigenvarieties associated to cuspidal Hilbert modular forms.

3.1 Complex Hilbert modular forms

Notation 3.1.1. (1) Let U be an open compact subgroup of $G(\mathbb{A}_f)$.

(2) In this section we set $J = \Sigma_{M_2(F)} = \Sigma_\infty$ (by Proposition 1.4.4, we can restrict to this case without loss of generality).

(3) Note that $C_\infty^+ = (\mathbb{R}^\times SO_2(\mathbb{R}))^{\Sigma_\infty}$ and we write the elements of $SO_2(\mathbb{R})^{\Sigma_\infty}$ as

$$u(\theta) = \begin{pmatrix} \cos(2\pi\theta_{\mathrm{v}}) & \sin(2\pi\theta_{\mathrm{v}}) \\ -\sin(2\pi\theta_{\mathrm{v}}) & \cos(2\pi\theta_{\mathrm{v}}) \end{pmatrix}_{\mathrm{v}\in\Sigma_\infty}$$

for some $\theta = (\theta_{\mathrm{v}})_{\mathrm{v}} \in \mathbb{R}^{\Sigma_\infty}$.

(4) Let $\mathfrak{I}_\infty$ be the F-modulus consisting of the product of all archimedean places of F.

(5) Let $\mathfrak{t}_i = t_i\widehat{\mathcal{O}}_F \cap F$ be as in 1.3.3.

Let us now unravel Definition 1.2.9 in this setting.

Definition 3.1.2. The $\mathbb{C}$-vector space of *cuspidal complex Hilbert modular forms* of weight (k, r) and level U, denoted $S_{k,r}(U)$ is given by functions $\mathbf{f} : G(\mathbb{A}) \to \mathbb{C}$ such that

(a) $\mathbf{f}(\gamma g u) = \mathbf{f}(g)$ for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$ and $u \in U$.

(b) For $u_\infty u(\theta) \in C_\infty^+$ and $g \in G(\mathbb{A})$ we have

$$\mathbf{f}(gu_\infty u(\theta)) = u_\infty^{-r} \exp\left(2\pi i \left(\sum_{\mathrm{v}\in\Sigma_\infty} k_{\mathrm{v}}\theta_{\mathrm{v}}\right)\right)\mathbf{f}(g).$$

(c) For each $x \in G(\mathbb{A}_f)$, the function f_x defined (as in Definition 1.2.9 (c)) is holomorphic in the variable z_{v} for all $\mathrm{v} \in \Sigma_\infty$.

(d) For all $g \in G_D(\mathbb{A})$ and for each additive Haar measure dx.

$$\int_{F\backslash\mathbb{A}_F} \mathbf{f}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0.$$

3.1.3 Fourier expansions

Since we are in the GL_2-case, our modular forms have q-expansions which are of great use when studying the action of Hecke operators. In order to give the q-expansions of Hilbert modular forms we will first decompose our spaces.

Definition 3.1.4. Let U be an open compact subgroup of $G(\mathbb{A}_f)$, with $\det(U) = \widehat{\mathcal{O}_F}^\times$ and let $\Gamma^i(U)$ be as in 1.3.4. Let (k, r, n, v, w) be a weight tuple. We define $S_{k,r}(\Gamma^i(U))$ to be the space of functions

$$f : \mathcal{H}^{\Sigma_\infty} \longrightarrow \mathbb{C}$$

such that

$$f(\gamma(z)) = j(\gamma, z)^k \det(\gamma)^{-w} f(z)$$

for all $\gamma \in \Gamma^i(U)$ and is holomorphic in z_{v} for all $\mathrm{v} \in \Sigma_\infty$. Moreover, we assume that f vanishes at *all* cusps of $\Gamma^i(U)$.

Proposition 3.1.5. *The map $f \mapsto (f_{t_i})_i$ with the notation as in Definition 1.2.9 (c), gives an isomorphism*

$$S_{k,r}(U) \cong \bigoplus_{i=1}^{h} S_{k,r}(\Gamma^i(U)).$$

Proof. See [Shi78b, Section 2]. □

We now give a special case of Proposition 3.1.5 which will be useful later on.

Corollary 3.1.6. *Let* $\Gamma_0(\mathfrak{b},\mathfrak{n}) = \mathrm{GL}_2(F)^+ \cap \begin{pmatrix} \mathcal{O}_F & \mathfrak{b}^* \\ \mathfrak{b}\mathfrak{n}\mathfrak{d}_F & \mathcal{O}_F \end{pmatrix}$ *and*

$$\Gamma_1(\mathfrak{b},\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{b},\mathfrak{n}) \mid d \equiv 1 \mod \mathfrak{n} \right\}$$

then

$$S_{k,r}(U_1(\mathfrak{n})) \cong \bigoplus_{i=1}^{h} S_{k,r}(\Gamma_1(\mathfrak{b}_i,\mathfrak{n})),$$

where $\mathfrak{b}_i^* = \mathfrak{t}_i$.

Notation 3.1.7. When needed we will denote elements of $S_{k,r}(U)$ as $\mathbf{f}$ and elements of $S_{k,r}(\Gamma^i(U))$ as f_i. Moreover, using the proposition above we write $\mathbf{f} = (f_1, \ldots, f_h)$ since $(f_1, \ldots, f_h)$ determines $\mathbf{f}$ by setting $\mathbf{f}(\gamma t_i u) = f_i|u_\infty(\mathbf{i})$ where $\gamma \in G(\mathbb{Q})$, $u \in U$ and $\mathbf{i} = (\sqrt{-1}, \ldots, \sqrt{-1})$.

Remark 3.1.8. It is important to note that the Hecke action preserves the spaces $S_{k,r}(U)$ but *not* the spaces $S_{k,r}(\Gamma^i(U))$. Specifically, given UxU for $x \in G(\mathbb{A})$ one can find, for each i, an element α_i such that $UxU = Ut_i^{-1}\alpha_i t_j U$ where j is uniquely determined by the condition that $\det(x)t_i t_j^{-1}\widehat{\mathcal{O}}_F \cap F$ is principal modulo $\mathfrak{I}_\infty$. If $\mathbf{f} = (f_1, \ldots, f_h)$, then $\mathbf{f}|[UxU] = (g_1, \ldots, g_h)$, where $g_j = f_i|[\Gamma^i(U)\alpha_i\Gamma^j(U)]$.

Let $U = U_*(\mathfrak{n})$ and let $\mathbf{f} \in S_{k,r}(U)$ with $\mathbf{f} = (f_1, \ldots, f_h)$. Now each f_i admits a Fourier expansion of the form

$$f_i(z) = \sum_{\xi \in \mathfrak{b}_i^+} c_i(\xi) e_F(\xi z),$$

where $\mathfrak{b}_i^* = \mathfrak{t}_i$, $e_F := e_\mathbb{Q} \circ \mathrm{Tr}_{F/\mathbb{Q}} : \mathbb{A}_F/F \to \mathbb{C}^*$, with $\mathrm{Tr}_{F/\mathbb{Q}}$ is the usual trace on the adeles and $e_\mathbb{Q}$ is the additive character determined by $e_\mathbb{Q}(x) = \exp(2\pi i x)$ if $x \in \mathbb{R}$, $\ker(e_\mathbb{Q}|_{\mathbb{Q}_l}) = \mathbb{Z}_l$ for l a prime and $e_\mathbb{Q}(q) = 1$ for $q \in \mathbb{Q}$.

Remark 3.1.9. Note that if $\epsilon \in \mathcal{O}_F^{\times,+}$, then $c_i(\epsilon\xi)\epsilon^w = c_i(\xi)$ if $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma^i(U)$ and $c_i(\epsilon^2\xi)\epsilon^k = c_i(\xi)$ if $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right) \in \Gamma^i(U)$. In particular, $c_i(\xi)\xi^v$ for $\xi \neq 0$ depends only on the ideal $\xi\mathcal{O}_F$ since $v + w = r + 1$ is parallel.

Definition 3.1.10. Let $\mathbf{f} = (f_i)_i$, then we can define a function on the group of fractional ideals by setting

$$\mathfrak{a}(\mathfrak{m}, \mathbf{f}) = \begin{cases} c_i(\xi)\xi^v & \text{if } \mathfrak{m} = \xi\mathfrak{t}_i \text{ and } \mathfrak{m} \text{ is integral,} \\ 0 & \text{otherwise.} \end{cases}$$

These will be our Fourier coefficients of our Hilbert modular forms $\mathbf{f}$. In fact, one has the following expansion (which follows from [Shi78a, (2.18)])

$$\mathbf{f}\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right) = \sum_{\xi \in F_+^\times} \mathfrak{a}(\xi y \mathcal{O}_F, \mathbf{f})(\xi y_\infty)^w e_F(\xi i y_\infty) e_F(\xi x).$$

We now define the Petersson inner product on these spaces.

Definition 3.1.11. Let $f, g \in S_{k,r}(\Gamma^i(U))$ then we define

$$\langle f, g \rangle = \mu(\Gamma^i(U) \backslash \mathcal{H}^g)^{-1} \int_{\Gamma^i(U)\backslash\mathcal{H}} \overline{f(z)} g(z) y^k d\mu(z)$$

where $z_\mathrm{v} = x_\mathrm{v} + i y_\mathrm{v}$, $d\mu(z) = \prod_{\mathrm{v} \in \Sigma_\infty} y_\mathrm{v}^{-2} dx_\mathrm{v} dy_\mathrm{v}$ and $\mu(\Gamma^i(U)\backslash\mathcal{H}^g)$ is the measure of a fundamental domain for $\Gamma^i(U)\backslash\mathcal{H}^g$ with respect to $d\mu(z)$. Moreover, for $\mathbf{f} = (f_i)_i, \mathbf{g} = (g_i)_i$ set

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{i=1}^{h} \langle f_i, g_i \rangle.$$

3.1.12 Level $U_0(\mathfrak{n})$ and $U_1(\mathfrak{n})$ Hilbert modular forms

We now restrict our level structure to $U_0(\mathfrak{n})$ and $U_1(\mathfrak{n})$ for $\mathfrak{n}$ and integral ideal. Note that $U_0(\mathfrak{n})/U_1(\mathfrak{n}) \cong (\mathcal{O}_F/\mathfrak{n})^\times$ therefore $U_0(\mathfrak{n})$ acts on $S_{k,r}(U_1(\mathfrak{n}))$ by $\mathbf{f} \mapsto \mathbf{f}|u$ for $u \in U_0(\mathfrak{n})$. If ψ (or $\psi_\mathfrak{n}$ if we want to keep track of the modulus) is a character of $(\mathcal{O}_F/\mathfrak{n})^\times$ extended to $U_0(\mathfrak{n})$ by setting $\psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(d_\mathfrak{n})$ (where $d_\mathfrak{n}$ is the image in $\mathcal{O}_\mathfrak{n}$ of d) and we let

$$S_{k,r}(U_0(\mathfrak{n}), \psi) = \{\mathbf{f} \in S_{k,r}(U_1(\mathfrak{n})) \mid \mathbf{f}(gu) = \psi(u)\mathbf{f}(g) \text{ for all } g \in G(\mathbb{A}), u \in U_0(\mathfrak{n})\}$$

then the following proposition is immediate.

Proposition 3.1.13. *There is a decomposition* $S_{k,r}(U_1(\mathfrak{n})) = \bigoplus_\psi S_{k,r}(U_0(\mathfrak{n}), \psi)$ *where the sum is over characters of* $(\mathcal{O}_F/\mathfrak{n})^\times$.

Remark 3.1.14. Note that if ψ is trivial then $S_{k,r}(U_0(\mathfrak{n}), \psi) = S_{k,r}(U_0(\mathfrak{n}))$.

Now, as we observed in Remark 3.1.8, the decomposition in Corollary 3.1.6 is not preserved by the Hecke action. To solve this problem we introduce Hecke characters Ψ of F extending $\psi_\mathfrak{n}$ with infinity type $-(r, \ldots, r)$, i.e., $\Psi(x) = x^{-r}$ for all $x \in F_\infty$ and $\Psi = \psi$ on $\hat{\mathcal{O}}_F$. Let

$$S_{k,r}(\mathfrak{n}, \Psi) = \{\mathbf{f} \in S_{k,r}(U_1(\mathfrak{n})) \mid \mathbf{f}(gz) = \Psi(z)\mathbf{f}(g) \text{ for all } g \in G(\mathbb{A}), z \in \mathbb{A}_F^*\}.$$

Then we have

$$S_{k,r}(U_1(\mathfrak{n})) = \bigoplus_{\Psi} S_{k,r}(\mathfrak{n}, \Psi)$$

where the index runs over Hecke characters Ψ with infinity type $-r$ and conductor dividing $\mathfrak{n}$ (of which there are only finitely many). Moreover, this decomposition *is* preserved by the Hecke operators $\boldsymbol{T}(U_1(\mathfrak{n}))$ see [Shi78a, Section 2] or [Dim03, Section 4.1].

Remark 3.1.15. It follows from the above that $S_{k,r}(U_0(\mathfrak{n}), \psi) = \bigoplus_{\Psi} S_{k,r}(\mathfrak{n}, \Psi)$ where the sum is over all Hecke characters Ψ extending ψ with infinity type $-r$.

3.2 Atkin-Lehner theory

We now study the action of Hecke operators on Hilbert modular forms of level $U_1(\mathfrak{n})$ and $U_0(\mathfrak{n})$. For this we begin by defining the relevant Hecke rings.

3.2.1 Hecke operators

For $U = U_0(\mathfrak{n})$ or $U_1(\mathfrak{n})$, let $\boldsymbol{T}(U, \Delta(\mathfrak{n}))$ be the Hecke ring as in Definition 1.4.2. For $\mathfrak{a} \subset \mathcal{O}_F$ and $\mathfrak{b} \nmid \mathfrak{n}$, define

$$T_{\mathfrak{a}} = \sum_{x \in D(\mathfrak{a})} [UxU] \qquad \text{and} \qquad S_{\mathfrak{b}} = \left[U \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) U\right]$$

where $D(\mathfrak{a}) = \{x \in \Delta(\mathfrak{n}) \mid \det(x)\mathcal{O}_F = \mathfrak{a}\}$ and $b\mathcal{O}_F = \mathfrak{b}$. Then one can show that $T_{\mathfrak{a}}, S_{\mathfrak{b}}$ generate $\boldsymbol{T}(U, \Delta(\mathfrak{n}))$, in fact, we can be more explicit: let v be a finite place of F and let π_{v} be a local uniformizer of $\mathcal{O}_{\mathrm{v}}$. Then for $\mathrm{v} \nmid \mathfrak{n}$ define $S_{\mathrm{v}} = \left[U \left(\begin{smallmatrix} \pi_{\mathrm{v}} & 0 \\ 0 & \pi_{\mathrm{v}} \end{smallmatrix}\right) U\right]$ and define $T_{\mathrm{v}} = \left[U \left(\begin{smallmatrix} \pi_{\mathrm{v}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U\right]$. If $\mathrm{w} | \mathfrak{n}$ we define $U_{\mathrm{w}} = \left[U \left(\begin{smallmatrix} \pi_{\mathrm{w}} & 0 \\ 0 & 1 \end{smallmatrix}\right) U\right]$. Then one can check that $\boldsymbol{T}(U, \Delta(\mathfrak{n}))$ is generated by $S_{\mathrm{v}}, T_{\mathrm{v}}, U_{\mathrm{w}}$ and for $\mathfrak{r}, \mathfrak{m}$ integral ideals

$$T_{\mathfrak{r}} T_{\mathfrak{m}} = \sum_{\mathfrak{r}+\mathfrak{m} \subset \mathfrak{a}} N_{F/\mathbb{Q}}(\mathfrak{a}) S_{\mathfrak{a}} T_{\mathfrak{a}^{-2}\mathfrak{r}\mathfrak{m}}$$

(cf. [Shi94, Chapter 3]).

Notation 3.2.2. Following Hida, in the case that the weight k is not parallel (i.e. $v \neq 0$), it is common to re-normalize the Hecke operator by setting $T_{\mathfrak{p}}^0 = \pi_{\mathfrak{p}}^{-v} T_{\mathfrak{p}}$ and $S_{\mathfrak{q}}^0 = \pi_{\mathfrak{q}}^{-2v} S_{\mathfrak{q}}$. This normalization is to ensure 'integrality' later on, but will not be needed for the results in this section.

We now explicitly write down the action of Hecke operators on the Fourier coefficients.

Notation 3.2.3. For Ψ a Hecke character of conductor $\mathfrak{n}\mathfrak{J}_\infty$, let Ψ^* be the ideal character defined modulo $\mathfrak{n}\mathfrak{J}_\infty$ such that $\Psi^*(\mathfrak{p}) = \Psi(\pi_\mathfrak{p})$ if $\mathfrak{p} \nmid \mathfrak{n}$ and is zero otherwise (even if Ψ is the trivial character).

Proposition 3.2.4. *Let* $\mathbf{f} \in S_{k,r}(\mathfrak{n}, \Psi)$*, then*

$$\mathfrak{a}(\mathfrak{r}, \mathbf{f}|T_\mathfrak{m}) = \sum_{\mathfrak{r}+\mathfrak{m}\subset\mathfrak{b}} \Psi^*(\mathfrak{b}) N_{F/\mathbb{Q}}(\mathfrak{b})^{r+1} \mathfrak{a}(\mathfrak{b}^{-2}\mathfrak{m}\mathfrak{r}, \mathbf{f}).$$

Proof. In order to make the calculation simpler we will only prove this in the special case that F has narrow class number one.[1] The extension of this case to general totally real fields is simple since by Remark 3.1.8 we can understand the action on $\mathbf{f}$ by understanding the action each of its components f_i. Let $\mathfrak{m} = m\mathcal{O}_F$ with m totally positive and, since we are in the class number one case, we can pick $t_1 = 1$. Then $\mathbf{f}(\gamma t_1 u) = f|u_\infty(\mathbf{i})$ for $\gamma \in G(\mathbb{Q})$, $u \in U$, so we can simply work with f. Then

$$f \mid T_\mathfrak{m} = \sum_{\substack{a,d \\ (ad)=\mathfrak{m} \\ (d,\mathfrak{n})=1}} \sum_{b\in(\mathcal{O}_F/d)} f| \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) =: \sideset{}{'}\sum_{b\in(\mathcal{O}_F/d)} f| \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$$

Now, the Fourier expansion of f is given by

$$f(z) = \sum_{\xi\in\mathcal{O}_F^*} c(\xi) \exp(2\pi i \operatorname{Tr}(\xi z)).$$

So, if we let

$$C(\xi) = m^{w+1-k} \sum_{\substack{d|m \\ \xi/d\in\mathcal{O}_F^*}} \Psi(d) d^{k-1} c(\xi m/d^2),$$

then

$$\begin{aligned} f|T_\mathfrak{m}(z) &= \sum_\xi c(\xi) \sideset{}{'}\sum_b \Psi(d) m^w d^{-k} \exp(2\pi i \operatorname{Tr}(\xi z a/d)) \exp(2\pi i \operatorname{Tr}(\xi b/d)) && (3.1) \\ &= \sum_\xi c(\xi) m^w \sum_{\substack{d|m \\ \xi/d\in\mathcal{O}_F^*}} \Psi(d) d^{1-k} \exp(2\pi i \operatorname{Tr}(\xi m z/d^2)) && (3.2) \\ &= \sum_{\xi'} C(\xi') \exp(2\pi i \operatorname{Tr}(\xi' z)) && (3.3) \end{aligned}$$

[1] Note that the general result is stated in [Shi78a, (2.23)].

where we note that (3.2) follows from the fact that

$$\sum_b \exp(2\pi i \operatorname{Tr}(\xi b/d)) = \begin{cases} N_{F/\mathbb{Q}}(d) = d^1 & \text{if } \xi/d \in \mathcal{O}_F^*, \\ 0 & \text{otherwise,} \end{cases}$$

and (3.3) follows from replacing d with m/d. The result then follows by observing that $\mathfrak{a}(\xi\mathcal{O}_F, f) = a(\xi)\xi^v$ and that $w+1-k=v$ and $k+2v-1=r+1$.

□

Remark 3.2.5. If one uses Notation 1.2.8, then $r+1 = k_0 - 1$ which in the above proposition we recover the more standard normalizations for this result.

Proposition 3.2.6. *Let $\langle -, - \rangle$ denote the Petersson inner product. Then, for $\mathbf{f}, \mathbf{g} \in S_{k,r}(\mathfrak{n}, \Psi)$ we have*

$$\Psi^*(\mathfrak{p})\langle \mathbf{f}|T_\mathfrak{p}, \mathbf{g}\rangle = \langle \mathbf{f}, \mathbf{g}|T_\mathfrak{p}\rangle \qquad \textit{for all } (\mathfrak{p}, \mathfrak{n}) = 1.$$

Proof. See [Shi78a, Proposition 2.4].

□

3.2.7. It follows at once that if $\mathbf{f} \in S_{k,r}(\mathfrak{n}, \Psi)$ is such that $\mathbf{f}|T_\mathfrak{m} = \lambda(\mathfrak{m})\mathbf{f}$ for $(\mathfrak{m}, \mathfrak{n}) = 1$ then $\lambda(\mathfrak{m}) = \Psi^*(\mathfrak{m})\overline{\lambda(\mathfrak{m})}$, where $\overline{a}$ denotes the complex conjugate of a. Moreover, note that the Hecke operators at an ideal not dividing the level will be normal (i.e. commute with their adjoints) from which it follows that they act semisimply, a fact which we will use later on.

3.2.8 Newforms and oldforms

We now want to define the *old* and *new* subspaces of Hilbert modular forms.

Definition 3.2.9. Let $\iota_\mathfrak{q} : S_{k,r}(\mathfrak{n}, \Psi) \to S_{k,r}(\mathfrak{n}\mathfrak{q}, \Psi)$ be such that $\mathfrak{a}(\mathfrak{m}, \iota_\mathfrak{q}(\mathbf{f})) = \mathfrak{a}(\mathfrak{m}\mathfrak{q}^{-1}, \mathbf{f})$ (recall that $\mathfrak{a}(\mathfrak{b}, \mathbf{f}) = 0$ if $\mathfrak{b}$ is not integral), then this property alone uniquely determines the operator. Furthermore, suppose the (finite part of the) conductor of Ψ, denoted $\mathfrak{c}_\Psi$, divides $\mathfrak{n}$, then we also have canonical injections $\iota : S_{k,r}(\mathfrak{n}, \Psi) \to S_{k,r}(\mathfrak{m}, \Psi)$.

Definition 3.2.10. Let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_F$ and Ψ a Hecke character of conductor $\mathfrak{c}_\Psi$ dividing $\mathfrak{n}$. We call the image of $S_{k,r}(\mathfrak{n}, \Psi)$ under $\iota_\mathfrak{q}$ and ι the $\mathfrak{q}$-*old* subspace of $S_{k,r}(\mathfrak{n}\mathfrak{q}, \Psi)$ and we denote it by

$$S_{k,r}^{\mathfrak{q}\text{-old}}(\mathfrak{n}\mathfrak{q}, \Psi).$$

With this we define $\mathfrak{q}$-*new* subspace

$$S_{k,r}^{\mathfrak{q}\text{-new}}(\mathfrak{n}\mathfrak{q}, \Psi)$$

to be the orthogonal complement (with respect to the Petersson inner product) of $S_{k,r}^{\mathfrak{q}\text{-old}}(\mathfrak{n}\mathfrak{q},\Psi)$ in $S_{k,r}(\mathfrak{n}\mathfrak{q},\Psi)$. Similarly, for $\mathfrak{m}$ any integral ideal of $\mathcal{O}_F$, we can define the subspace of $\mathfrak{m}$-*old* and $\mathfrak{m}$-*new* forms, by setting

$$S_{k,r}^{\mathfrak{m}\text{-old}}(\mathfrak{n}\mathfrak{m},\Psi) = \bigoplus_{\mathfrak{q}|\mathfrak{m}} S_{k,r}^{\mathfrak{q}\text{-old}}(\mathfrak{n}\mathfrak{m},\Psi),$$

and defining the space of $\mathfrak{m}$-*new* forms to be its orthogonal complement. Lastly, we denote by $S_{k,r}^{\text{old}}(\mathfrak{n},\Psi)$ the subspace of $S_{k,r}(\mathfrak{n},\Psi)$ generated by $\iota_{\mathfrak{q}}(\mathbf{f})$ for $\mathbf{f}\in S_{k,r}(\mathfrak{b},\Psi)$ for all $\mathfrak{b}$ such that $\mathfrak{c}_\Psi \mid \mathfrak{b}$ and $\mathfrak{b}\mid\mathfrak{n}$ $(\mathfrak{b}\neq\mathfrak{n})$ and $\mathfrak{q}$ runs over all divisors of $\mathfrak{b}^{-1}\mathfrak{n}$ and $S_{k,r}^{\text{new}}(\mathfrak{n},\Psi)$ is its orthogonal complement.

Before moving on to the Atkin-Lehner theory we first state a result which will be useful later on.

Proposition 3.2.11. *Let* $(\mathfrak{n},\mathfrak{p})=1$. *The matrix for the action of* $U_{\mathfrak{p}}$ *on* $S_{k,r}^{\mathfrak{p}\text{-}old}(\mathfrak{n}\mathfrak{p},\Psi)$ *is given by*

$$\begin{pmatrix} T_{\mathfrak{p}} & 1 \\ -\Psi^*(\mathfrak{p})N(\mathfrak{p})^{r+1} & 0 \end{pmatrix}.$$

Proof. First note that the map

$$\begin{aligned} S_{k,r}(\mathfrak{n},\Psi)^2 &\longrightarrow S_{k,r}^{\mathfrak{p}\text{-old}}(\mathfrak{n}\mathfrak{p},\Psi) \\ (f,g) &\longmapsto \iota(f)+\iota_{\mathfrak{p}}(g) \end{aligned}$$

induces an isomorphism of Hecke modules. The result then follows by noting that $\iota_{\mathfrak{p}}(g)|U_{\mathfrak{p}} = \iota(g)$ and using Proposition 3.2.4. □

We now recall some 'Atkin-Lehner type' results for Hilbert modular forms.

Definition 3.2.12. We call $\mathbf{f}\in S_{k,r}^{\text{new}}(\mathfrak{n},\Psi)$ a *primitive* newform if $\mathfrak{a}(\mathcal{O}_F,\mathbf{f})=1$ and it is a simultaneous eigenform for all $T_{\mathfrak{p}}$ for $\mathfrak{p}$ a prime ideal with $(\mathfrak{p},\mathfrak{n})=1$.

Theorem 3.2.13. *Let* $\mathfrak{b}\subset\mathcal{O}_F$ *be a fixed integral ideal and* $\mathbf{f}\in S_{k,r}(\mathfrak{n},\Psi)$ *be such that* $\mathfrak{a}(\mathfrak{m},\mathbf{f})=0$ *for all* $(\mathfrak{m},\mathfrak{b})=1$ *then* $\mathbf{f}\in S_{k,r}^{old}(\mathfrak{n},\Psi)$.

Proof. This is stated in [SW93, Theorem 3.1], but the actual proof is a generalization of [Li75, Section 2, Theorem 2 and Corollary 1]. □

Corollary 3.2.14. *If* $\mathbf{f}\in S_{k,r}^{new}(\mathfrak{n},\Psi)$ *then* $\mathbf{f}$ *is uniquely determined (up to a scalar multiple) by its eigenvalues.*

Proof. This follows at once by noting that if $\mathbf{f}$ and $\mathbf{g}$ are both eigenforms for $T_{\mathfrak{p}}$ for all $\mathfrak{p}$ prime with $(\mathfrak{p},\mathfrak{n})=1$ then $\mathbf{f}-\mathbf{g}\in S_{k,r}^{\text{old}}(\mathfrak{n},\Psi)$. □

In fact, we have a much stronger version of this given by the following theorem:

Theorem 3.2.15. *Let $\mathbf{f} \in S_{k,r}(\mathfrak{n}, \Psi)$ and $\mathbf{g} \in S_{k,r}(\mathfrak{m}, \Phi)$ be primitive newforms and assume that they have the same eigenvalues for all $T_{\mathfrak{b}}$ with $(\mathfrak{b}, \mathfrak{n}\mathfrak{m}) = 1$ then $\Psi = \Phi$, $\mathfrak{n} = \mathfrak{m}$ and $\mathbf{f} = \mathbf{g}$.*

Proof. This is [SW93, Theorem 3.6]. □

For any $\mathfrak{p}$ prime ideal with $(\mathfrak{p}, \mathfrak{n}) = 1$ the Hecke operators $T_{\mathfrak{p}}$ preserve $S^{\text{new}}_{k,r}(\mathfrak{n}, \Psi)$, therefore one can find a orthogonal basis of eigenforms for the $T_{\mathfrak{p}}$. Moreover, using Proposition 3.2.4 one gets, as in the case of modular forms over $\mathbb{Q}$, that if $\mathbf{f}$ is a primitive newform, then $\mathbf{f}|T_{\mathfrak{p}} = \mathfrak{a}(\mathfrak{p}, \mathbf{f})\mathbf{f}$ for all $(\mathfrak{p}, \mathfrak{n}) = 1$.

Theorem 3.2.16. *Let $\mathbf{f} \in S_{k,r}(\mathfrak{n}, \Psi)$ be a primitive newform and let Ψ have conductor $\mathfrak{c}$. Then for $\mathfrak{p} \mid \mathfrak{n}$ we have:*

(a) If $\mathfrak{n}_{\mathfrak{p}} = \mathfrak{c}_{\mathfrak{p}}$ then $\mid \mathfrak{a}(\mathfrak{p}, \mathbf{f}) \mid = N_{F/\mathbb{Q}}(\mathfrak{p})^{(k_0-1)/2}$.

(b) If $\mathfrak{n}_{\mathfrak{p}} = \mathfrak{p}$ and $\mathfrak{c}_{\mathfrak{p}} = 1$ then $\mathfrak{a}(\mathfrak{p}, \mathbf{f})^2 = \Psi(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})^{(k_0-2)}$.

(c) If $\mathfrak{p}^2 \mid \mathfrak{n}$ and $\mathfrak{n}_{\mathfrak{p}} \neq \mathfrak{c}_{\mathfrak{p}}$ then $\mathfrak{a}(\mathfrak{p}, \mathbf{f}) = 0$.

Here $(-)_{\mathfrak{p}}$ denotes the $\mathfrak{p}$ component of the ideal.

Proof. This is [SW93, Theorem 3.3]. □

Let $\mathfrak{n} = \mathfrak{r}\mathfrak{s}$ with $(\mathfrak{r}, \mathfrak{s}) = 1$. Let Ψ be a Hecke character whose associated Dirichlet character is $\psi_{\mathfrak{n}}$ and write $\psi_{\mathfrak{n}} = \psi_{\mathfrak{r}}\psi_{\mathfrak{s}}$. We now define (following [SW93]) the Atkin-Lehner involution.

Definition 3.2.17. Let $\Psi_{\mathfrak{r}}$ be a Hecke character extending $\psi_{\mathfrak{r}}$ (with infinity type $-r$) and choose $y \in G(\mathbb{A}_f)$ with $y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \begin{pmatrix} \mathfrak{r} & \mathcal{O}_F^* \\ \mathfrak{n}\mathfrak{d}_F & \mathfrak{r} \end{pmatrix}$ and $\det(y)\mathcal{O}_F = \mathfrak{r}$. Then define

$$W_{\mathfrak{r}}(\Psi_{\mathfrak{r}}) : S_{k,r}(\mathfrak{n}, \Psi) \to S_{k,r}(\mathfrak{n}, \Psi\overline{\Psi}_{\mathfrak{r}}^2)$$

by

$$\mathbf{f}|W_{\mathfrak{r}}(\Psi_{\mathfrak{r}})(x) = \overline{\Psi}_{\mathfrak{r}}(\det(x))\overline{\psi_{\mathfrak{r}}}(bt_{\mathfrak{d}_F})\overline{\psi_{\mathfrak{s}}}(a)\mathbf{f}(xy^{\iota}),$$

where $t_{\mathfrak{d}_F}\mathcal{O}_F = \mathfrak{d}_F$ and $y^{\iota} = \det(y)y^{-1}$.

This operator has many very useful properties, one of which is that it sends newforms to newforms. These results along with many others can be found in [SW93]. Lastly, we state a result which will be useful later when studying slopes.

Theorem 3.2.18. *Let $\mathfrak{r} \mid \mathfrak{n}$ with $(\mathfrak{r}, \mathfrak{n}\mathfrak{r}^{-1}) = 1$ and let $\mathbf{f} \in S_{k,r}(\mathfrak{n}, \Psi)$ be a primitive newform with $\mathbf{f} \mid W_{\mathfrak{r}}(\Psi_{\mathfrak{r}}) = \lambda(\mathbf{f})\mathbf{g}$ and $\mathbf{g} \in S_{k,r}(\mathfrak{n}, \Psi\overline{\Psi}_{\mathfrak{r}}^2)$ a primitive newform. Then*

(a) For $\mathfrak{p}$ a prime ideal,

$$\mathfrak{a}(\mathfrak{p}, \mathbf{g}) = \begin{cases} \overline{\Psi}_{\mathfrak{r}}^*(\mathfrak{p})\mathfrak{a}(\mathfrak{p}, \mathbf{f}) & \text{if } \mathfrak{p} \nmid \mathfrak{r}, \\ (\Psi\Psi_{\mathfrak{r}}^{-1})^*(\mathfrak{p})\overline{\mathfrak{a}(\mathfrak{p}, \mathbf{f})} & \text{if } \mathfrak{p} \mid \mathfrak{r}. \end{cases}$$

(b) $\mid \lambda(\mathbf{f}) \mid = 1$.

Again with notation as in 3.2.3.

Proof. This is [SW93, Theorem 4.2 and Lemma 4.3]. □

3.3 Geometric Hilbert modular forms

In this section we will (following [AIP16b]) reformulate the definitions of the spaces of Hilbert modular forms in a more geometric way which naturally generalizes to the p-adic setting and is the basis for defining overconvergent Hilbert modular cusp forms.

To define the spaces of Hilbert modular forms for $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$, we first work with the group $G^* = G \times_{\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m$ where $G \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ is given by the determinant morphism and $\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ is the natural diagonal morphism. We will define the spaces of modular forms for G^* and then, using a projector, one gets the definition for G. The reason for working with G^* is that the relevant moduli problem associated to G^* is representable while the one for G is not. We will begin by defining these moduli problems and then show how to define the relevant spaces of modular forms.

3.3.1 Abelian varieties with real multiplication

Definition 3.3.2. Let S be a scheme. An *abelian scheme* $A_{/S}$ is a proper, smooth and geometrically irreducible[2] group scheme over S. If $S = \mathrm{Spec}(K)$ is a point, then $A_{/S}$ is called an abelian variety over K.

It is not immediately clear that abelian schemes are commutative group schemes, but this follows from the rigidity lemma.

Proposition 3.3.3. *If $A_{/S}$ is an abelian scheme then it is a commutative group scheme.*

Proof. See [MFK94, Corollary 6.5]. □

[2]This means that its fibre at every geometric point of S is irreducible.

Definition 3.3.4. Let $A^\vee$ be the connected component of the identity in $\mathrm{Pic}_{A_{/S}}$. This is called the ***dual abelian scheme*** of A.

Remark 3.3.5. The fact that $A^\vee$ is even a scheme is non-trivial in general. See [FC90, Chapter I].

Definition 3.3.6. A ***polarization*** of $A_{/S}$ is a homomorphism $\lambda : A \to A^\vee$ such that for each geometric point s of S, the induced map $\lambda_s : A_s \to A_s^\vee$ is of the form $\Phi_{\mathcal{L}} : a \mapsto T_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$ for $\mathcal{L}$ some ample invertible sheaf on A_s, where $T_a(b) = a + b$.

Definition 3.3.7. Let F be a totally real field of degree g. An ***abelian variety with real multiplication (AVRM$_{/S}$)*** by a totally real field F is an abelian scheme $A \to S$ of relative dimension g together with an embedding of algebras $\iota_A : \mathcal{O}_F \hookrightarrow \mathrm{End}(A/S)$.

Definition 3.3.8. Let $\mathfrak{c}$ be a fractional ideal in F. For $A_{/S}$ an AVRM, one can define a sheaf of $\mathcal{O}_F$-modules on the big étale site of S which associates to a S-scheme Y the $\mathcal{O}_F$-module $A(Y) \otimes_{\mathcal{O}_F} \mathfrak{c}$. This functor is representable[3] by an AVRM which is denoted $A \otimes_{\mathcal{O}_F} \mathfrak{c}$ and is characterized by

$$A \otimes_{\mathcal{O}_F} \mathfrak{c} = \begin{cases} A/A[\mathfrak{c}^{-1}] & \text{if } \mathfrak{c}^{-1} \text{ is integral,} \\ (A^\vee \otimes \mathfrak{c}^{-1})^\vee & \text{if } \mathfrak{c} \text{ is integral.} \end{cases}$$

Moreover, we note that ι_A induces a map $\mathfrak{c} \hookrightarrow \mathrm{Hom}_{\mathcal{O}_F}(A, A \otimes_{\mathcal{O}_F} \mathfrak{c})$.

Definition 3.3.9. Let

$$\mathrm{Sym}_{\mathcal{O}_F}(A, A^\vee) = \{\lambda : A \to A^\vee \mid \lambda = \lambda^\vee, \lambda \circ \iota_A(r) = \iota_A(r)^\vee \circ \lambda \text{ for all } r \in \mathcal{O}_F\}$$

and let $P(A)$ be the cone of polarizations in $\mathrm{Sym}_{\mathcal{O}_F}(A, A^\vee)$. A ***Hilbert-Blumenthal abelian variety over S (HBAV$_{/S}$)*** is an AVRM $A_{/S}$ such that there exists a $\mathcal{O}_F$-equivariant homomorphism $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ inducing $(\mathfrak{c}, \mathfrak{c}^+) \cong (\mathrm{Sym}_{\mathcal{O}_F}(A, A^\vee), P(A)))$. Such a λ is called a $\mathfrak{c}$-***polarization***. If $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^\vee$ is an isomorphism, we say A satisfies the ***Deligne-Pappas condition***.

Remark 3.3.10. If the discriminant of F in invertible in S then the Deligne–Pappas condition is equivalent to the ***Rapoport condition*** which asks that for $\pi : A \to S$ the sheaf $\pi_*\left(\Omega^1_{A/S}\right)$ is (Zariski) locally free of rank 1 over $\mathcal{O}_S \otimes \mathcal{O}_F$. See [Gor02, Chapter 3.5].

Definition 3.3.11. Following [Hid04], we consider the following fibered category $\mathcal{A}_F$, whose objects are triples $(A, \iota_A, \lambda)_{/S}$ where A is a HBAV$_{/S}$ with a $\mathfrak{c}$-polarization λ, real

[3] See (for example), [Dim03, Section 3.1].

multiplication given by ι_A and the fibre functor is $(A, \iota_A, \lambda)_{/S} \mapsto S$. The morphisms are given by $\mathcal{O}_F$-linear morphisms $f : A_{/S} \to A'_{/S}$ of abelian schemes with $\lambda = f^\vee \circ \lambda' \circ f$.

Let us now consider the case $S = \mathrm{Spec}(\mathbb{C})$. Following [Gor02, Chapter 2, Section 2], we have the following example of an abelian variety with real multiplication by $\mathcal{O}_F$.

Definition 3.3.12. Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of $\mathcal{O}_F$. For $\tau \in \mathcal{H}^g$ let

$$\Lambda_\tau = \mathfrak{a} \cdot \tau + \mathfrak{b} = \{a_1\tau_1 + b_1, \ldots, a_g\tau_g + b_g \mid a \in \mathfrak{a}, b \in \mathfrak{b}\} \subset \mathbb{C}^g$$

where a_i, b_i denote the images of a, b under the corresponding embedding of F into $\mathbb{C}$. For $f \in F$ and $c = (c_1, \ldots, c_g) \in \mathbb{C}^g$ we let $f \cdot c = (f_1c_1, \ldots, f_gc_g)$. This then defines a complex torus $A_\tau = \mathbb{C}^g/\Lambda_\tau$ with real multiplication by $\mathcal{O}_F$.

Since we are over $\mathbb{C}$, defining a polarization on A_τ is equivalent to giving a real alternation form on Λ_τ (see [Gor02, Chapter 1, Section 6.1]).

Definition 3.3.13. Let $r \in (\mathfrak{a}\mathfrak{b})^*$. Then define $E_r : \mathfrak{a} \oplus \mathfrak{b} \times \mathfrak{a} \oplus \mathfrak{b} \to \mathbb{Z}$ by

$$E_r((x_1, y_1), (x_2, y_2)) = \mathrm{Tr}_{F/\mathbb{Q}}(r(x_1y_2 - x_2y_1)).$$

Now, define $E_{r,\tau} : \Lambda_\tau \times \Lambda_\tau \to \mathbb{Z}$ by setting $E_{r,\tau}((a\tau + b), (a'\tau + b') = E_r((a, b), (a', b'))$.

One can then show that (see [Gor02, Chapter 2, Corollary 2.10]) $E_{r,\tau}$ defines a polarization on A_τ if and only if $r \in ((ab)^*)^+$.

3.3.14 Hilbert-Blumenthal moduli

In this section we will study the following moduli problem:

Definition 3.3.15. Let (A, ι, λ) in $\mathcal{A}_F$ be as in Definition 3.3.11 (so λ is a $\mathfrak{c}$-polarization). Let $\mathfrak{n}$ be a non-zero ideal and let $\mu_\mathfrak{n}$ denote the locally free group scheme of finite rank given by $\mu_\mathfrak{n}(R) = \{x \in \mathbb{G}_m(R) \otimes_\mathbb{Z} \mathfrak{d}_F^{-1} | \mathfrak{n}x = 0\}$. Let $\mathfrak{n} \cap \mathbb{Z} = (N)$ and let $M(\mathfrak{c}, \mu_\mathfrak{n})$ be the Hilbert moduli scheme representing the functor

$$\mathcal{E}_{\mu_\mathfrak{n}} : \mathrm{Sch}\,/_{\mathbb{Z}[1/N]} \longrightarrow \mathrm{Set}$$

where $\mathcal{E}_{\mu_\mathfrak{n}}(S)$ is the set of isomorphism classes of $(A_{/S}, \iota, \lambda, \Phi_\mathfrak{n})$. Here $\Phi_\mathfrak{n} : \mu_\mathfrak{n} \hookrightarrow A[N]_{/S}$ is a closed immersion compatible with $\mathcal{O}_F$-actions. We call such a $\Phi_\mathfrak{n}$ a $\mu_\mathfrak{n}$-*level structure* on A.

Remark 3.3.16. If we take $\mu_{(N)}$ with $N \geq 3$, then the associated moduli problem is representable by a scheme $M(\mathfrak{c}, \mu_{(N)})$ (cf. [Gor02, Chapter 3, Theorem 6.9]).

Let us now consider the $\mathbb{C}$ points of $M(\mathfrak{c}, \mu_{\mathfrak{n}})$. These parametrize quadruples $(A, \iota, \lambda, \Phi_{\mathfrak{n}})$, where A is an abelian variety over $\mathbb{C}$ with real multiplication by $\mathcal{O}_F$ given by ι, a $\mathfrak{c}$-polarization and a $\mu_{\mathfrak{n}}$-level structure. We now have the following result: let

$$\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})^+ = \left\{ \gamma \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{a}^{-1}\mathfrak{b} \\ \mathfrak{a}\mathfrak{b}^{-1} & \mathcal{O}_F \end{pmatrix} \mid \det(\gamma) \in \mathcal{O}_F^{\times,+} \right\}$$

act on $\mathcal{H}^g$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \left(\frac{a_i\tau_i + b_i}{c_i\tau_i + d_i}\right)_i$.

Theorem 3.3.17. *(1) The isomorphism classes of $(A, \iota)/\mathbb{C}$ such that there exists $\mathfrak{c}$ polarisation λ is parametrized by $\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})^+ \backslash \mathcal{H}^g$, where $(ab)^* = \mathfrak{c}$.*

(2) The isomorphism classes of $(A, \iota)/\mathbb{C}$ with a given $\mathfrak{c}$ polarisation λ is parametrized by $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})^+ \backslash \mathcal{H}^g$, where $(ab)^* = \mathfrak{c}$ and $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})^+$ is the subgroup of $\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})^+$ of matrices with determinant 1.

Proof. This is [Gor02, Chapter 2, Theorem 2.17]. □

By fixing a set of representatives of $\mathrm{Cl}(F)^+$ of the form $(\mathfrak{c}, \mathfrak{c}^+)$ one can show that:

Corollary 3.3.18. *(1) There is a natural bijection between isomorphism classes of $(A, \iota)/\mathbb{C}$ and*

$$\coprod_{(\mathfrak{c}, \mathfrak{c}^+)} \mathrm{GL}(\mathcal{O}_F \oplus \mathfrak{c})^+ \backslash \mathcal{H}^g$$

and $(A, \iota)/\mathbb{C}$ is parametrized by $\mathrm{GL}(\mathcal{O}_F \oplus \mathfrak{c})^+$ if and only if there exists some $\mathfrak{c}^*$-polarisation on A.

(2) There is a natural bijection between isomorphism classes of (A, ι, λ) and

$$\coprod_{(\mathfrak{c}, \mathfrak{c}^+)} \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{c})^+ \backslash \mathcal{H}^g.$$

Proof. f This is [Gor02, Chapter 2, Corollary 2.19]. □

3.3.19. We now need to take into account the $\mu_{\mathfrak{n}}$-level structure. Let $N = \mathfrak{n} \cap \mathbb{Z}$, and let μ_N be the set of N-th roots of unity, which we identify with $N^{-1}\mathbb{Z}/\mathbb{Z}$ via the exponential map. This then induces an isomorphism $\mu_{(N)} = \mu_N \otimes \mathfrak{d}_F^{-1} \cong N^*/\mathcal{O}_F^*$ which further induces $\mu_{\mathfrak{n}} \cong \mathfrak{n}^*/\mathcal{O}_F^*$. So to give a $\mu_{\mathfrak{n}}$-level structure on A_τ we define an inclusion $\Phi_{\mathfrak{n}} : \mathfrak{n}^*/\mathcal{O}_F^* \hookrightarrow A_\tau$ defined by $\Phi_{\mathfrak{n},\tau}(j \mod \mathcal{O}_F^*) = j + \Lambda_\tau$. So if we want to parametrize quadruples $(A, \iota, \lambda, \Phi_{\mathfrak{n}})/\mathbb{C}$ we need to find the subgroup of matrices in $\mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{c})^+$ that preserve this level structure. Now, an easy check shows that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{Q})^+$ with $\gamma(\tau') = \tau$, multiplication by $c\tau' + d = (c_i\tau_i' + d_i)_i$

induces an isogeny $(A_\tau, \iota, \lambda, \Phi_{\mathfrak{n},\tau}) \to (A_{\tau'}, \iota, \det(\gamma)\lambda, (c\tau' + d)\Phi_{\mathfrak{n},\tau'})$. Therefore, if we set $\Gamma_1^1(\mathfrak{c}, \mathfrak{n}) = \Gamma_1(\mathfrak{c}, \mathfrak{n}) \cap \mathrm{SL}_2(F)$ (with notation as in 3.1.6), we see that $\Gamma_1^1(\mathfrak{c}, \mathfrak{n})\backslash\mathcal{H}^g$ parametrizes isomorphism classes of $(A, \iota, \lambda, \Phi_{\mathfrak{n}})/\mathbb{C}$ where λ is a $\mathfrak{c}$-polarization. It follows that, for $\mathfrak{n}$ sufficiently small, $M(\mu_n, \mathfrak{c})(\mathbb{C}) = \Gamma_1^1(\mathfrak{c}, \mathfrak{n})\backslash\mathcal{H}^g$.

Theorem 3.3.20. *There is a natural bijection between isomorphism classes of* $(A, \iota, \lambda, \Phi_{\mathfrak{n}})$ *and*

$$\coprod_{(\mathfrak{c},\mathfrak{c}^+)} \Gamma_1^1(\mathfrak{c}, \mathfrak{n})\backslash\mathcal{H}^g.$$

Proof. This follows from Corollary 3.3.18 and 3.3.19. □

If instead we consider the moduli problem associated to the functor

$$\mathcal{E}_{\mu_{\mathfrak{n}}} : \mathrm{Sch}\,/_{\mathbb{Z}[1/N]} \longrightarrow \mathrm{Set}$$

given by letting $\mathcal{E}_{\mu_{\mathfrak{n}}}(S)$ be the set of isomorphism classes of $(A_{/S}, \iota, \overline{\lambda}, \Phi_{\mathfrak{n}})$, where $\overline{\lambda}$ is a polarization class.[4] One can then show that for $\mathfrak{n}$ small enough, there is a coarse moduli scheme $M^G(\mathfrak{c}, \mu_{\mathfrak{n}})$ representing $\mathcal{E}_{\mu_{\mathfrak{n}}}$ (see [Hid04, Theorem 4.5]). In fact, over $\mathbb{C}$ the quadruples are parametrized by $\Gamma_1(\mathfrak{c}, \mathfrak{n})\backslash\mathcal{H}^g$ and using the above, one has:

Theorem 3.3.21. *There is a natural bijection between isomorphism classes of* $(A, \iota, \overline{\lambda}, \Phi_{\mathfrak{n}})$ *and*

$$\coprod_{(\mathfrak{c},\mathfrak{c}^+)} \Gamma_1(\mathfrak{c}, \mathfrak{n})\backslash\mathcal{H}^g = Y(U_1(\mathfrak{n}))$$

with $U_1(\mathfrak{n})$ *as in 1.3.7.*

Proof. This is analogous to Theorem 3.3.20. This can also be deduced from [TX16, Proposition2.4]. □

3.3.22 Geometric Hilbert modular forms for G^* and G

From now on $\mu_{\mathfrak{n}}$ will be a level structure (as above) for which the associated moduli problem is representable.

Notation 3.3.23. Let $M(\mathfrak{c}, \mu_{\mathfrak{n}})$ denote the scheme representing the corresponding moduli problem (for G^*). Denote by $\overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}})$ (resp. $\overline{M}^*(\mathfrak{c}, \mu_{\mathfrak{n}})$) *a* fixed *toroidal* (resp. *the minimal*) compactification of $M(\mathfrak{c}, \mu_{\mathfrak{n}})$.

[4]Here we say that $(A_{/S}, \iota, \overline{\lambda}, \Phi_{\mathfrak{n}}) \cong (A'_{/S}, \iota', \overline{\lambda}', \Phi'_{\mathfrak{n}})$ if there is an isomorphism $f : A \to A'$ with $f \circ \Phi_{\mathfrak{n}} = \Phi'_{\mathfrak{n}}$ and $\overline{\lambda} = \overline{f^*\lambda'}$.

Definition 3.3.24. Let $\boldsymbol{A}$ be the universal semiabelian scheme over $\overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}})$, with real multiplication by $\mathcal{O}_F$. Let $e : M(\mathfrak{c}, \mu_{\mathfrak{n}}) \to \boldsymbol{A}$ be the identity section and define

$$\omega_{\boldsymbol{A}} = e^* \left(\Omega^1_{\boldsymbol{A}/M(\mathfrak{c},\mu_{\mathfrak{n}})} \right)$$

to be the conormal sheaf of $\boldsymbol{A}$.

Remark 3.3.25. By the Rapoport condition there exists a greatest open subscheme $\overline{M}^R(\mathfrak{c}, \mu_{\mathfrak{n}}) \subset \overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}})$ such that $\omega_{\boldsymbol{A}}$ is an invertible $\mathcal{O}_{\overline{M}^R(\mathfrak{c},\mu_{\mathfrak{n}})} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module.

We will use $\omega_{\boldsymbol{A}}$ to define *another* invertible sheaf whose sections will be our Hilbert modular forms. This sheaf will be associated to a classical weight k, from which we can then define the spaces of Hilbert modular forms for G^* of weight k (and appropriate level). Since we will later be interested in constructing space of overconvergent Hilbert modular forms, which are p-adic objects, we will define the spaces over a p-adic field (although this construction can be done over $\mathbb{C}$ which recovers the definitions in Section 3.1).

Definition 3.3.26. We define a classical algebraic weight for G^* as a map from $\mathbb{T}(\mathbb{Z}_p)$ to $\mathbb{C}_p$ defined by an element $k \in \mathbb{Z}^g_{\geq 0}$, as usual.

Definition 3.3.27. Let k be a classical weight for G^*. Then, define the invertible modular sheaf

$$\Omega^k := \bigotimes_{\mathrm{v} \in \Sigma_\infty} \omega_{\boldsymbol{A},\mathrm{v}}^{\otimes k_{\mathrm{v}}},$$

where $\omega_{\boldsymbol{A},\mathrm{v}} := \omega_{\boldsymbol{A}} \otimes_{\mathrm{v}} \mathcal{O}_L$ and here $\otimes_{\mathrm{v}}$ denotes the tensor over $\mathcal{O}_L \otimes \mathcal{O}_F$ via $1 \otimes \iota_{\mathrm{v}}$.

Definition 3.3.28. The L-vector space of $\mathfrak{c}$-polarized, tame level $\Gamma^1_1(\mathfrak{c}, \mathfrak{n})$ and weight k *Hilbert modular forms for G^** is defined by

$$M_k(\Gamma^1_1(\mathfrak{c}, \mathfrak{n})) := H^0(\overline{M}^R(\mathfrak{c}, \mu_{\mathfrak{n}}), \Omega^k).$$

The subspace of cusp forms is defined by

$$S_k(\Gamma^1_1(\mathfrak{c}, \mathfrak{n})) := H^0(\overline{M}^R(\mathfrak{c}, \mu_{\mathfrak{n}}, \Omega^k(-B)),$$

where $B := \overline{M}(\mathfrak{c}, \mu_{\mathfrak{n}}) \backslash M(\mathfrak{c}, \mu_{\mathfrak{n}})$ is the boundary divisor in the toroidal compactification.

To define the spaces of Hilbert modular forms associated to G, we need to introduce a certain projector. First, we note that, by 3.3.19, multiplication by $\epsilon \in \mathcal{O}_F^{\times,+}$ gives an isomorphism $(A, \iota, \lambda, \Phi) \cong (A, \iota, \epsilon^2\lambda, \epsilon\Phi)$. Now, let $\mathfrak{S}_{\mathfrak{n}}$ be the elements of $\mathcal{O}_F^{\times,+}$ congruent to 1 modulo $\mathfrak{n}$. Define an action of $\mathfrak{O} := \mathcal{O}_F^{\times,+}/\mathfrak{S}^2_{\mathfrak{n}}$ on $M(\mathfrak{c}, \mu_{\mathfrak{n}})$, by

$$\epsilon \cdot (A, \iota, \lambda, \Phi) := (A, \iota, \epsilon\lambda, \Phi).$$

Since multiplication by ϵ gives an isomorphism $\epsilon : A \to A$ such that $\epsilon^* \lambda = \epsilon^2 \lambda$, it follows that, if $\epsilon = \eta^2 \in \mathfrak{S}_{\mathfrak{n}}$, then ϵ acts trivially on $(A, \iota, \lambda, \Phi)$; hence the action factors through $\mathfrak{O}$ as required.

Definition 3.3.29. Let (k, r, n, v, w) be a weight tuple with (k, r) a classical algebraic weight (for G). We define an action of $\mathfrak{O}$ on Ω^k by sending a local section f of Ω^k on $M^R(\mathfrak{c}, \mu_{\mathfrak{n}})$ to

$$(\epsilon \cdot f) : (A, \iota, \lambda, \Phi, \beta) \to w(\epsilon) f(A, \iota, \epsilon^{-1}\lambda, \Phi, \beta)$$

where $\epsilon \in \mathcal{O}_F^{\times,+}$ and β is a local generator for $\omega_{\boldsymbol{A}}$ as a $\mathcal{O}_{M^R(\mathfrak{c},\mu_{\mathfrak{n}})} \otimes \mathcal{O}_F$-module. If $\epsilon = \eta^2 \in \mathfrak{S}_{\mathfrak{n}}$, then this acts trivially. Hence the action factors through $\mathfrak{O}$. With this we define a *projector*

$$\mathfrak{e}_{k,r} : M_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n})) \to M_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}))$$

by

$$\mathfrak{e}_{k,r} := \frac{1}{\mid \mathfrak{O} \mid} \sum_{\epsilon \in \mathfrak{O}} \epsilon.$$

We can now define the space of Hilbert modular cusp forms for G.

Definition 3.3.30. The L-vector space of *classical Hilbert modular forms for G of level* $\Gamma_1(\mathfrak{n}, \mathfrak{c})$*, and weight* (k, r) is defined to be the image of $\mathfrak{e}_{k,r}$ and is denoted $M^G_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$. Similarly, we let $S^G_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ be the image of $S_k(\Gamma_1^1(\mathfrak{c}, \mathfrak{n}))$ under $\mathfrak{e}_{k,r}$.

We now have a similar situation as before, in that these spaces will not be fixed by the Hecke operators. In fact, note that $F^{\times,+}$ acts on the pairs $(\mathfrak{c}, \mathfrak{c}^+)$ by $\epsilon(\mathfrak{c}, \mathfrak{c}^+) = (\epsilon\mathfrak{c}, \epsilon\mathfrak{c}^+)$, which induces an isomorphism $\alpha_\epsilon : M^G_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n})) \to M^G_{k,r}(\Gamma_1(\epsilon\mathfrak{c}, \mathfrak{n}))$. Moreover, if $\epsilon \in \mathcal{O}_F^{\times,+}$, then $\alpha_\epsilon(f) = f$ for all $f \in M^G_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$.

Definition 3.3.31. We define the space of *classical Hilbert modular forms for G of level* $U_1(\mathfrak{n})$ *and weight* (k, r) denoted $M^G_{k,r}(U_1(\mathfrak{n}))$ as V/I where

$$V := \bigoplus_{(\mathfrak{c}, \mathfrak{c}^+)} M^G_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$$

and $I = (f - \alpha_\epsilon(f))_{\epsilon \in \left(F^{\times,+}/\mathcal{O}_F^{\times,+}\right)}$. We define $S^G_{k,r}(U_1(\mathfrak{n}))$ similarly.

On $M^G_{k,r}(U_1(\mathfrak{n})), S^G_{k,r}(U_1(\mathfrak{n}))$ one can define Hecke operators as in [Hid04, Section 4.1.10].

Remark 3.3.32. We note here that there are other ways of defining Hilbert modular forms for G as sections of a sheaf $\Omega^{k,r}$ on $M^G(\mathfrak{c}, \mu_{\mathfrak{n}})$ (cf. [TX16, Section 2.2]). Working over $\mathbb{C}$ one then recovers the spaces $S_{k,r}(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ as defined in Corollary 3.1.6. To see the relationship with our definition one observes that there is a morphism $m : M(\mathfrak{c}, \mu_{\mathfrak{n}}) \to M^G(\mathfrak{c}, \mu_{\mathfrak{n}})$ which is finite and Galois, with Galois group $\mathfrak{D}$ such that $\Omega^{k,r} = (m_*(\Omega^k))^{\mathfrak{D}}$ (cf. [AIP16b, Section 1]).

3.4 Overconvergent Hilbert modular forms

In this section we will give a short overview of the constructions of spaces of overconvergent Hilbert modular forms together with some of their properties. The details of the construction as well as proofs can be found in [AIP16b].

3.4.1 The weight space

The weight space is a rigid analytic variety that allows us make precise the notion of modular forms 'living' in p-adic families. We will also define the weight space for G^* and G and show how they are related.

Definition 3.4.2. We define $\mathcal{W}^G$ to be the rigid analytic space over L associated to the completed group algebra $\mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]$. We call $\mathcal{W}^G$ the *weight space for* G. Moreover, let

$$[-] : \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \longrightarrow \mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times]\!]^\times$$

denote the *universal character* of $\mathcal{W}^G$.

3.4.3. It follows from the above definition that the weight space $\mathcal{W}^G$ is the rigid analytic space over L, representing the functor sending any L-algebra A to $\mathrm{Hom}_{cts}(\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times, A^\times)$. Moreover, we note that $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times \cong H \times \mathbb{Z}_p^{g+1}$, where H is the torsion subgroup of $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$. From this it follows that

$$\mathcal{W}^G \cong H^\vee \times B(1,1)^{g+1} \cong \bigsqcup_{\chi \in H^\vee} \mathcal{W}_\chi$$

as rigid spaces, where $H^\vee$ is the character group of H and $B(1,1)$ is the open ball of radius 1 around 1. It is clear from this that $\mathcal{W}^G$ is equidimensional of dimension $g+1$.

Notation 3.4.4. Elements of $\mathcal{W}^G(\mathbb{C}_p)$ will be given by $v : \mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ and $r : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. Setting $n = -2v + r$ and $\kappa = n + 2$, we will continue to denote these weights as (κ, r) and call (κ, r, n, v, w) a weight tuple if κ, r, n, v, w satisfy the same relations as in Definition 1.2.7. More generally, if $\mathfrak{U}$ is an affinoid with a morphism of rigid spaces $\mathfrak{U} \to \mathcal{W}^G$, then we will denote by $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ the restriction of the universal character to $\mathfrak{U}$.

Definition 3.4.5. Let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$ a classical algebraic weight. This defines an *algebraic weight* by sending $(a, b) \in \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ to $a^v b^r$.

Notation 3.4.6. There is a natural map $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ given by

$$t \mapsto (t^{-2}, N_{F/\mathbb{Q}}(t)). \tag{3.4}$$

In this way we view weights $(\kappa, r) \in \mathcal{W}^G$ with (κ, r, n, v, w) a weight tuple as maps $\mathbb{T}(\mathbb{Z}_p) \to \mathbb{C}_p$ given by $t \mapsto n(t)$.

Definition 3.4.7. *The weight space* $\mathcal{W}^{G^*}$ is defined by setting $\mathcal{W}^{G^*}$ to be the rigid analytic space over L associated to $\mathcal{O}_L[\![\mathbb{T}(\mathbb{Z}_p))]\!]$, where $\mathbb{T}$ and L are as before. There is a canonical map $\mathcal{W}^G \to \mathcal{W}^{G^*}$ induced from (3.4).

Notation 3.4.8. Let τ denote the Teichmüller character, and for $s \in \mathbb{Z}_{\geq 0}^g$ we let τ^s be the character of $\mathbb{T}(\mathbb{Z}_p)_{tors}$ which is τ^{s_i} on the i-th component.

Definition 3.4.9. A weight $(\kappa, r) \in \mathcal{W}^G$ is called *arithmetic* or *classical* if it is the product of a algebraic character and a finite character ψ. We will denote such weights by $(\kappa, r)\psi$ or simply as (κ_ψ, r) with the understanding that we require κ, r to both be algebraic. We will usually let ψ be a character of $\mathcal{O}_L$ of conductor dividing p^s, viewed as a character of $\mathbb{T}(\mathbb{Z}_p)$ (via strong approximation).

Remark 3.4.10. In the literature there are slightly more general weight spaces than the one we have introduced. One alternative way of defining the weight space is to let $\mathcal{W}'$ denote the rigid analytic space associated to the completed group algebra $\mathcal{O}_L[\![T(\mathbb{Z}_p)]\!]$, where T is the standard maximal torus of G. The problem with this weight space is that it contains too many weights for which the associated spaces of modular forms would be empty. For this reason one usually imposes suitable vanishing conditions on these weights. See [Buz07, Part III] and [Urb11, 4.3.2]. The weight spaces one gets this way conjecturally have dimension $g+1$ (dependent on Leopoldt's conjecture). For this reason we have chosen to work with $\mathcal{W}^G$ which has the correct dimension. Moreover, if Leopoldt's conjecture is true then the resulting eigenvarieties for the different weight spaces will be isomorphic.

We will later want to study the geometric structure of the associated eigenvarieties. For this, we define here the centre and boundary of the weight space. We begin by thinking of the weight space as an adic space. In this setting, one defines (following [AIP16a]) $\mathcal{W}_{adic} = \mathrm{Spa}(\Lambda_F, \Lambda_F)^{an}$, where $\Lambda_F = \Lambda_F^0[H]$ with $\Lambda_F^0 = \mathcal{O}_L[\![T_1, \ldots, T_g]\!]$.[5] To see what the boundary should be, we can restrict to the trivial component of the weight

[5]Note that here, for consistency, we are defining the weight space over $\mathcal{O}_L$, but with more care one can work over $\mathbb{Z}_p$ which is more customary when discussing integral models, see [AIP16a, Section 2], but we do not need this here.

space, i.e., $\mathcal{W}^0 = \mathrm{Spa}(\Lambda_F^0, \Lambda_F^0)^{an}$, where Λ_F^0 has the $(p, T_1, \dots, T_g)$-adic topology (here p is assumed unramified in F).

Definition 3.4.11. Now define a continuous map (cf. [Sch14, Proposition 3.3.5]) $c : \mathcal{W}^0 \longrightarrow [0,\infty]^g$ by

$$x \longmapsto \left(\frac{\log |T_1(\tilde{x})|}{\log |p(\tilde{x})|}, \dots, \frac{\log |T_1(\tilde{x})|}{\log |p(\tilde{x})|} \right),$$

where $\tilde{x}$ is the maximal generalization of x. Note that $\log |T_i(\tilde{x})|$ and $\log |p(\tilde{x})|$ take values in $[-\infty, 0)$ since the T_i and p are topologically nilpotent. From this it follows that $c(x) = (0, \dots, 0)$ if and only if $|p(\tilde{x})| = 0$. Moreover, we note that we cannot have x such that only some of the entries of $c(x)$ are zero, i.e., we cannot have $c(x) = (0, x_2, \dots, x_g)$ with $x_i \neq 0$. With this set-up, being near ***boundary of the weight space*** (in this component) is the same as having a point $x \in \mathcal{W}^0$ with $c(x)$ close to zero.

As an example of weights that are near the boundary, we can take a classical weight (k_ψ, r) where ψ is a character sufficiently ramified at ***every*** prime above p. Now a natural question is, what if we take ψ a character only ramified at ***some*** of the primes above p? It is not clear to the author if these points should morally be in the boundary of the weight space of in the "centre", for this reason we define a quasi-boundary (which contains the boundary) as follows:

Definition 3.4.12. Let $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_g)$ be a weight on $\mathbb{T}(\mathbb{Z}_p) \cong H \times \mathbb{Z}_p^g$. Fox a fixed choice of $h \in H$, i.e. for a fixed component, let γ_i be a topological generator of the i-th copy of $\mathbb{Z}_p$. Then, define $w(\kappa) = (\kappa_i(\gamma_i) - 1) \in \mathbb{C}_p^g$. In this way we obtain a coordinate in the weight space for each of our weights. We also set $\mathrm{val}_p(w(\kappa)) = \min_i\{\mathrm{val}_p(\kappa_i(\gamma_i) - 1)\}$ and say that for an odd prime p (resp. $p = 2$), a weight κ is near the ***quasi-boundary*** if $\mathrm{val}_p(w(\kappa)) \leq 1$ (resp. $\mathrm{val}_2(w(\kappa)) < 3$), otherwise we say it is near the ***centre***.

Later, when defining the spaces of locally analytic functions it will be convenient for us to extend the definition of the weight space from $\mathbb{T}$ to T, which denotes the maximal torus of G. We do this as follows:

Definition 3.4.13. Let (κ, r, n, v, w) be a weight tuple with $(\kappa, r) \in \mathcal{W}^G$ and set

$$\lambda_{\kappa,r} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \lambda_1(a)\lambda_2(d)$$

where $\lambda_1 = (r+n)/2$, $\lambda_2 = (r-n)/2$.

Remark 3.4.14. Note that if we map $T(\mathbb{Z}_p)$ to $\mathbb{T}(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ via $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mapsto (a/d, \mathrm{Norm}(a))$ then our weights on T and $\mathbb{T}$ agree.

Remark 3.4.15. Using this, we talk about weights λ on T where we implicitly assume that there is some $(\kappa, r) \in \mathcal{W}^G$ such that $\lambda = \lambda_{\kappa,r}$. This construction then lets us take a weight $\mathcal{W}^G$ and get a weight in $\mathcal{W}'$ as in Remark 3.4.10.

3.4.16 Overconvergent spaces

Our goal is now to associate to each weight (κ, r) or family of weights $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$, a space of overconvergent Hilbert modular forms. There are several constructions of these spaces but we will be interested in the construction given by [AIP16b]. In this case, one defines an overconvergent sheaf which interpolates the sheaf Ω^k from Definition 3.3.27. Using this, one can then define the spaces of overconvergent Hilbert modular forms for G^* and then, using a projector, define the spaces for G. The construction of the overconvergent sheaf can be found in [AIP16b, AIP15, AIS14, Hat16], so we only give some of its properties.

Let f be the number of primes above p in F and let $t_m \in \mathbb{Q}^f$ be a multi-index with $0 < t_i \leq \frac{1}{p^m}$ for $m \geq 1$. Let $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_\mathfrak{n})$ and $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_\mathfrak{n})$ denote the formal completions of $\overline{M}(\mathfrak{c}, \mu_\mathfrak{n}), \overline{M}^*(\mathfrak{c}, \mu_\mathfrak{n})$ along their special fibres. Now, let $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}), \overline{\mathcal{M}}^*(\mathfrak{c}, \mu_\mathfrak{n})$ denote the rigid fibres of $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_\mathfrak{n}), \overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_\mathfrak{n})$ respectively and let $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m), \overline{\mathcal{M}}^*(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$ denote the neighbourhoods of the respective ordinary locus defined by the condition that $\mathrm{val}_p(h_{\mathfrak{p}_i}) \cap [0, 1] \leq t_i$, where $h_{\mathfrak{p}_i}$ are the ***partial Hasse invariants*** as defined in [AIP16b, 3.2.1].

Now, the overconvergent sheaves are defined over formal models of $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$ and $\overline{\mathcal{M}}^*(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$, which are obtained as follows: let $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m))$ (resp. $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_\mathfrak{n}, t_m))$ be the normalization of the formal model of $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$ (resp. $\overline{\mathcal{M}}^*(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$) given by taking iterated blow-ups along the ideals $(h_{\mathfrak{p}_i}, p^{t_i})$ of $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_\mathfrak{n})$ (resp. $\overline{\mathfrak{M}}^*(\mathfrak{c}, \mu_\mathfrak{n})$) and removing all divisors at infinity. Then on $\overline{\mathfrak{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m))$ we can construct the following sheaves:

Theorem 3.4.17 (Andreatta-Iovita-Pilloni). *For every m-analytic weight $(\kappa, r) \in \mathcal{W}^{G^*}(L)$ there exists a coherent sheaf $\Omega^{\dagger,(\kappa,r)}$ of $\mathcal{O}_{\overline{\mathfrak{M}}(\mathfrak{c},\mu_\mathfrak{n},t_m)}$-modules whose restriction to the rigid analytic fibre $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$ is invertible.*

More generally, to each affinoid $\mathfrak{U}$ with a morphism $\mathfrak{U} \to \mathcal{W}^{G^*}$ and m such that $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ is locally m-analytic, one can attach a coherent sheaf $\Omega^{\dagger,(\kappa^{\mathfrak{U}},r^{\mathfrak{U}})}$ of $\mathcal{O}_{\overline{\mathfrak{M}}(\mathfrak{c},\mu_\mathfrak{n},t_m)\times\hat{\mathfrak{U}}}$-modules where $\hat{\mathfrak{U}} = \mathrm{Spf}(A)$ is the formal model of $\mathfrak{U}$, where A consists of power bounded elements of $\mathfrak{U}$. Moreover, the restriction of $\Omega^{\dagger,(\kappa^{\mathfrak{U}},r^{\mathfrak{U}})}$ to the rigid fibre is invertible. Lastly, if (k, r) is a classical weight, then $\Omega^{\dagger,(k,r)}$ agrees on $\overline{\mathcal{M}}(\mathfrak{c}, \mu_\mathfrak{n}, t_m)$ with the classical $\Omega^{(k,r)}$ as in Remark 3.3.32.

Proof. See [AIP16b, Sections 3.4-3.5]. □

Remark 3.4.18. In general, the $\Omega^{\dagger,(\kappa^{\mathfrak{U}},r^{\mathfrak{U}})}$ and $\Omega^{\dagger,(\kappa,r)}$ depend on m, but when restricted to the rigid fibres, they are independent of m, for this reason we have suppressed the dependence on m. See [AIP16b, Proposition 3.9 and Proposition 3.13].

Using these sheaves, one can then define the spaces of t_m-overconvergent cuspidal Hilbert modular forms for G^* of weight $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ by setting

$$S_{\mathfrak{U}}^{\dagger}(\Gamma_1^1(\mathfrak{c},\mathfrak{n}),t_m) = \mathrm{H}^0(\overline{\mathcal{M}}(\mathfrak{c},\mu_{\mathfrak{n}},t_m)\times\mathfrak{U},\Omega^{\dagger,(\kappa^{\mathfrak{U}},r^{\mathfrak{U}})}(-B))$$

where B is again the boundary divisor. From this, one then uses a projector to define families of t_m-overconvergent cuspidal Hilbert modular forms for G of weight $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ denoted $S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t_m)$. Moreover, taking $\mathfrak{U}=\mathrm{Spf}(L)$ gives $S_{\kappa,r}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t_m)$.

Theorem 3.4.19. *Let $\mathfrak{U}$ be an admissible open affinoid of $\mathcal{W}^G$ and $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ as in 3.4.4. Let A be the algebra of power bounded elements of $\mathfrak{U}$. Then for an appropriate*[6] *choice of m and t_m the spaces $S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t_m)$ are Banach $(A\otimes_{\mathcal{O}_L}L)$-modules satisfying (Pr). Moreover, for any weight $(\kappa,r)\in\mathfrak{U}(L)$ there is a natural specialization map*

$$S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t_m)\longrightarrow S_{\kappa,r}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t_m)$$

which is surjective.

Proof. This is [AIP16b, Theorem 4.4]. □

As before, these spaces have an action of $F^{\times,+}$, so they will not be fixed under the action of Hecke operators. In particular, we have:

Lemma 3.4.20. *Let $\epsilon\in F^{\times,+}$ and assume that ϵ is also a p-adic unit. Then there is a canonical isomorphism*

$$\mathscr{L}_\epsilon : S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t)\longrightarrow S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t)$$

which only depends on ϵ modulo totally positive units.

Proof. This is [AIP16b, Lemma 4.5]. □

Definition 3.4.21. Let

$$S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}),t) := \left(\bigoplus_{\mathfrak{c}\in Frac(F)^{(p)}} S_{\mathfrak{U}}^{G,\dagger}(\Gamma_1(\mathfrak{c},\mathfrak{n}),t)\right)/\,(\mathscr{L}_\epsilon(f)-f)_{\epsilon\in Princ(F)^{+,(p)}}$$

[6]This means such that $\kappa^{\mathfrak{U}}$ is m-analytic (see Definition 4.2.2)

be the Banach module of tame level U, *t-overconvergent cuspidal arithmetic Hilbert modular forms* for G with weights parametrized by $\mathfrak{U}$. Here $Frac(F)^{(p)}$ is the group of fractional ideals prime to p and $Princ(F)^{+,(p)}$ is the group of positive elements which are p-adic units.

Moreover, taking the limit over t we get Frechet spaces $S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}))$ which give a quasi-coherent sheaf of overconvergent cuspidal arithmetic Hilbert modular forms $S^{G,\dagger}(U_1(\mathfrak{n}))$ over $\mathcal{W}^G$, whose value at an open affinoid $\mathfrak{U} \subset \mathcal{W}^G$ is $S_{\mathfrak{U}}^{G,\dagger}(U_1(\mathfrak{n}))$.

Remark 3.4.22. Note that taking $\mathfrak{U} = \mathrm{Sp}(L)$ with image (κ, r) in $\mathcal{W}^G$ will give the spaces of this fixed weight.

Following [AIP16b, Section 4.3], for $\mathfrak{q}$ prime to the tame level, one can define commuting Hecke operators $T_{\mathfrak{q}}, S_{\mathfrak{q}}$ action on $S^{G,\dagger}(U_1(\mathfrak{n}))$. Moreover, for $\mathfrak{p}|p$ one can define operators $U_{\mathfrak{p}}$ such that $U_p = \prod_{\mathfrak{p}|p} U_{\mathfrak{p}}^{e(\mathfrak{p})}$ for $e(\mathfrak{p})$ the ramification degree of $\mathfrak{p}$.

Proposition 3.4.23. *The U_p operator is a compact operator on $S_{\kappa,r}^{G,\dagger}(U_1(\mathfrak{n}))$ for any weight (κ, r).*

Proof. This follows from [AIP16b, Lemma 3.27]. □

Definition 3.4.24. Let $h \in \mathbb{Q}_{\geq 0}$. We say an element $f \in S_{\kappa,r}^{G,\dagger}(U_1(\mathfrak{n}))$ has *slope-*$\leq h$ for U_p (resp. $U_{\mathfrak{p}}$ for $\mathfrak{p}|p$) if it is annihilated by a unitary polynomial in U_p (resp. $U_{\mathfrak{p}}$) whose roots have valuation less than h.

Remark 3.4.25. Note that if f is in fact an eigenform, then having slope-$\leq h$ for U_p (resp. $U_{\mathfrak{p}}$) is saying that the p-adic valuation of the U_p (resp. $U_{\mathfrak{p}}$) eigenvalue is less than h.

Notation 3.4.26. For each $\mathrm{v} \in \Sigma_\infty$, we have a field embedding ι_{v} of F into $\mathbb{C}$ given by v; this map extends to a map $F_p \longrightarrow \overline{\mathbb{Q}}_p$ and then factors through the projection $F_p \longrightarrow F_{\mathfrak{p}}$ for some $\mathfrak{p}$ above p. This then gives a natural surjection $\Sigma_\infty \to \Sigma_p$ where $\mathrm{v} \mapsto \mathfrak{p}_{\mathrm{v}}$. For each prime ideal $\mathfrak{p} \in \Sigma_p$ let $\Sigma_{\mathfrak{p}}$ be the set of $\mathrm{v} \in \Sigma_\infty$ factoring through the projection $F_p \to F_{\mathfrak{p}}$.

Definition 3.4.27. Let (k, r, n, v, w) be a weight tuple with (k, r) be a classical algebraic weight. For each prime ideal $\mathfrak{p}_j \in \Sigma_p$ we define $v_{\mathfrak{p}}(k, r) = \sum_{i \in \Sigma_{\mathfrak{p}}} v_i$. If (k, r) is any classical weight, we define $v_{\mathfrak{p}}$ by considering the algebraic part of the weight.

Theorem 3.4.28 (Control Theorem). *Let (k, r) be a classical weight in $\mathcal{W}^G$. Let $f \in S_{k,r}^{G,\dagger}(U)$ be a finite slope (for U_p) overconvergent Hilbert modular form whose $U_{\mathfrak{p}_i}$ slope is less than h_i for $\mathfrak{p}_i \in \Sigma_p$. If p is unramified and $h_i < v_{\mathfrak{p}_i}(k, r) + \min_{j \in \Sigma_{\mathfrak{p}_i}} \{k_j - 1\}$ for all i, then f is a classical form.*

Proof. See [TX16, Theorem 1]. □

We now wish to use Buzzard's Eigenmachine to construct the eigenvariety of Hilbert modular forms. One of the key ingredients is the existence of links which is checked explicitly in [Hat16, Section 3.3.3].

Theorem 3.4.29. *Associated to the eigendata of* $(\mathcal{W}^G, S^{G,\dagger}(U_1(\mathfrak{n})), \boldsymbol{T}, U_p)$ *we have an eigenvariety* $\mathscr{X}_G(U_1(\mathfrak{n}))$ *with the following properties:*

(a) *It is equidimensional of dimension* $g+1$.

(b) *There is a universal character* $\phi : \boldsymbol{T} \to \mathscr{O}_{\mathscr{X}}$.

(c) *There is a map* $\alpha : \mathscr{X} \to \mathcal{W}^G$ *that is locally on* $\mathscr{X}$ *and* $\mathcal{W}^G$*, finite and surjective.*

(d) *For all* $(\kappa, r) \in \mathcal{W}^G$*, the points* $\alpha^{-1}(\kappa, r)$ *are in bijection with the finite slope eigensystems occurring in* $S^{G,\dagger}(U_1(\mathfrak{n})) \mid_{\kappa,r} = S^{G,\dagger}_{\kappa,r}(U_1(\mathfrak{n}))$.

Proof. This is [AIP16b, Theorem 5.1]. □

Chapter 4

Totally definite quaternionic modular forms

Following [Buz07, Part III], we will define classical and overconvergent modular forms on a totally definite quaternion algebra D over F and prove the control theorem in this setting. In contrast to [Buz07], we will work with the weight space $\mathcal{W}^G$ which has the advantage of being equidimensional of dimension $g+1$ (recall $[F:\mathbb{Q}]=g$). Apart from this small detail, the rest of our construction spaces of overconvergent quaternionic modular forms over F follows [Buz07, Part III]. Throughout this chapter our chosen prime p may be ramified unless otherwise stated.

4.1 Classical spaces

We will define the spaces of classical quaternionic modular forms using a definition that, compared Section 1.2 , is more suited to p-adic interpolation; the crucial difference being that the actions are 'shifted' from the infinite places to the places above p (cf. 1.2.4).

Notation 4.1.1. (1) Let D/F be a totally definite quaternion algebra split above p. Note that this means we have an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_p \cong M_2(\mathcal{O}_p)$, where $\mathcal{O}_D$ is the maximal order of D and $\mathcal{O}_p := \mathcal{O}_F \otimes \mathbb{Z}_p$. This then induces an isomorphism $D_p := G_D(F_p) = D \otimes_F F_p \cong M_2(F_p)$.

(2) Let $\pi_\mathfrak{p}$ denote the uniformisers of $F_\mathfrak{p}$ and $\pi \in \mathcal{O}_p$ be the element whose $\mathfrak{p}$ component of $\mathcal{O}_p$ is $\pi_\mathfrak{p}$. For $s \in \mathbb{Z}^{\Sigma_p}$ we let $\pi^s = (\pi_\mathfrak{p}^{s_\mathfrak{p}})$. By abuse of notation we also let π denote the ideal of $\mathcal{O}_F$ which is the product of all the prime ideals above p, i.e., the radical of $p\mathcal{O}_F$.

(3) Let $G_D/\mathbb{Q}_p := \mathrm{Res}_{F/\mathbb{Q}}(D^\times) \times \mathbb{Q}_p$, which is a connected reductive linear algebraic

group over $\mathbb{Q}_p$ (via our choice of splitting). Let T be the standard maximal torus, B the standard Borel subgroup, and the unipotent radical N. We denote by $\overline{B}$ and $\overline{N}$ be the opposite Borel and opposite unipotent radical. Let $I \subset G_D(\mathbb{Z}_p)$ be the standard Iwahori subgroup in good position with respect to B (in good position means that $B, N, T, G_D, \overline{N}$ have fixed compatible integral models over $\mathbb{Z}_p$).

(4) For $m \in \mathbb{Z}_{\geq 1}^{\Sigma_p}$, set

$$I_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Z}_p) \mid c \in \pi^m \mathcal{O}_p \right\},$$

with $I = I_1 = I_{(1,\dots,1)}$ and let $\overline{I}_m = \overline{N}(\mathbb{Z}_p) \cap I_m$.

Furthermore, we set

$$T^+ = \left\{ t \in T(\mathbb{Q}_p) \mid tN(\mathbb{Z}_p)t^{-1} \subseteq N(\mathbb{Z}_p) \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T(\mathbb{Q}_p) \mid ab^{-1} \in \mathcal{O}_p \right\}$$

and

$$T^{++} = \left\{ t \in T(\mathbb{Q}_p) \mid \bigcap_{i>0} t^i N(\mathbb{Z}_p) t^{-i} = \{1\} \right\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T(\mathbb{Q}_p) \mid ab^{-1} \in p\mathcal{O}_p \right\}.$$

With this we define the semigroup $\Delta = IT^+I$. Note that the Iwahori decomposition tells us that

$$I = \overline{I}_1 T(\mathbb{Z}_p) N(\mathbb{Z}_p),$$

and hence any $\delta \in \Delta$ can be written uniquely as $\delta = \overline{n}_\delta t_\delta n_\delta$ with $\overline{n}_\delta \in \overline{I}_1, t_\delta \in T^+, n_\delta \in N(\mathbb{Z}_p)$.

Definition 4.1.2. Let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty} \times \mathbb{Z}$. Let V_k be the L-vector space with basis of monomials $\prod_{\mathrm{v}\in\Sigma_\infty} Z_{\mathrm{v}}^{m_{\mathrm{v}}}$, with $m \in \mathbb{Z}_{\geq 0}^{\Sigma_\infty}$, $0 \leq m_{\mathrm{v}} \leq k_{\mathrm{v}} - 2$. We define a right action of $\Delta = IT^+I$ on this space as follows: for $\gamma = (\gamma_{\mathfrak{p}})_{\mathfrak{p}\in\Sigma_p} = \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix}_{\mathfrak{p}} \in \Delta$, let

$$\gamma : \prod_{\mathrm{v}\in\Sigma_\infty} Z_s^{m_s} \longmapsto \prod_{\mathrm{v}\in\Sigma_\infty} (c_{\mathrm{v}} Z_{\mathrm{v}} + d_{\mathrm{v}})^{n_{\mathrm{v}}} \det(\gamma_{\mathrm{v}})^{v_{\mathrm{v}}} \left(\frac{a_{\mathrm{v}} Z_{\mathrm{v}} + b_{\mathrm{v}}}{c_{\mathrm{v}} Z_s + d_{\mathrm{v}}} \right)^{m_{\mathrm{v}}}$$

Note that here (following [Buz07]) we have adopted the notation that for $a_{\mathfrak{p}}$ (resp. $b_{\mathfrak{p}}, c_{\mathfrak{p}}, d_{\mathfrak{p}}$) we let a_{v} (resp. $b_{\mathrm{v}}, c_{\mathrm{v}}, d_{\mathrm{v}}$) denote the image of $a_{\mathfrak{p}}$ under the corresponding map $\iota \circ \iota_{\mathrm{v}}$ for $\mathrm{v} \in \Sigma_{\mathfrak{p}}$ as described in 3.4.26.

Let $V_{n,v}(L)$ denote the resulting Δ-module.[1]

Definition 4.1.3. Let U be an open compact subgroup of $G_D(\mathbb{A}_f)$, such that its image under the projection $U \to D_p^\times$ lies in I_m for some $m \in \mathbb{Z}_{\geq 0}^{\Sigma_p}$ with $m \geq 1 = (1, \ldots, 1)$ (with the natural ordering) and let (k, r, n, v, w) be a weight tuple with $(k, r) \in \mathcal{W}^G$ a classical weight. The space of ***quaternionic modular forms*** over D of weight k and level U, denoted $S_{k,r}^D(U)$, is the space of functions

$$f : G_D(\mathbb{A}_f) \longrightarrow V_{n,v}(L)$$

such that:

(a) For $\gamma \in G_D(\mathbb{Q})$, we have $f(\gamma g) = f(g)$ for all $g \in G_D(\mathbb{A})$.

(b) For $u \in U$ we have $f(g) = f(gu^{-1}) \cdot u_p$ for all $g \in G_D(\mathbb{A})$, where u_p denotes the p-part of u.

Remark 4.1.4. By choosing a field homomorphism $L \to \mathbb{C}$ one can base change this construction to $\mathbb{C}$ and again give them an action of U at infinity. The resulting spaces will be isomorphic to those defined in 1.2.9, with $\Sigma_D = \emptyset$.

4.2 Overconvergent spaces

We are now going to define the spaces of overconvergent modular forms for D, which interpolate the classical spaces. For this we need to find a larger Δ-module containing $V_{n,v}(L)$, so we work with the spaces of locally analytic functions.

4.2.1 Locally analytic functions

Let $X \subset \mathbb{Q}_p^s$ be open and compact.

Definition 4.2.2. For a finite extension $L/\mathbb{Q}_p$, we say a function $f : X \to L$ is L-***analytic*** if it can be expressed as a converging power series

$$f(x_1, \ldots, x_s) = \sum_{t_1, \ldots, t_s} \alpha_{t_1,\ldots,t_s}(x_1 - a_1)^{t_1} \cdots (x_s - a_s)^{t_s},$$

for $\alpha_{t_1,\ldots t_s} \in L$, and some $(a_1, \ldots, a_s) \in X$. We say it is ***algebraic*** if almost all α's are zero.

[1]Note that since we have chosen weights such that $n + 2v$ is parallel, $\mathcal{O}_F^{\times,+}$ will act trivially, when embedded diagonally into I via $\mathcal{O}_F \to \mathcal{O}_p \to M_2(\mathcal{O}_p)$.

Definition 4.2.3. For $m \in \mathbb{Z}_{\geq 0}^r$, let $\mathcal{A}(X, L, m)$ be the L-vector space of *m-locally analytic functions*, i.e., functions that are analytic on balls of radius p^{-m} covering X. Then $\mathcal{A}_m(X, L)$ is a p-adic Banach space when X is compact. We let

$$\mathcal{A}(X, L) = \bigcup_{m \geq 0} \mathcal{A}(X, L, m).$$

This is the space of functions $f : X \to L$ that are m-locally L-analytic for some n.

We now define the Δ-modules that we will be interested in.

Definition 4.2.4. We begin by identifying $\mathcal{O}_p$ with an open compact subset of $\mathbb{Q}_p^g$ compatible with the identification of I as an open compact of $\mathbb{Q}_p^{4g}$. We then consider

$$\mathcal{A}(\mathcal{O}_p, L) = \bigcup_{m \geq 0} \mathcal{A}(\mathcal{O}_p, L, m).$$

This is a Δ-module with the following action. For (κ, r, n, v, w) a weight tuple with $(\kappa, r) \in \mathcal{W}^G(L)$, $f \in \mathcal{A}(\mathcal{O}_p, L)$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$ and $z \in \mathcal{O}_p$, let

$$(f \cdot \gamma)(z) = n(cz + d)v(\det(\gamma))f\left(\frac{az + b}{cz + d}\right).$$

We denote this module by $\mathcal{A}_{n,v}(\mathcal{O}_p, L)$.

Lemma 4.2.5. *For $(\kappa, r) \in \mathcal{W}^G$ there exists a smallest $m(\kappa, r)$, such that for all $m \geq m(\kappa, r)$, (κ, r) is m-locally analytic.*

Proof. See [Urb11, Lemma 3.2.5]. □

From this it follows that

$$\mathcal{A}_{n,v}(\mathcal{O}_p, L) = \bigcup_{m \geq m(\kappa, r)} \mathcal{A}_{n,v}(\mathcal{O}_p, L, m),$$

where $\mathcal{A}_{n,v}(\mathcal{O}_p, L, m)$ is the Δ-module $\mathcal{A}(\mathcal{O}_p, L, m)$ with the action defined as above. More generally, since we wish to consider families of modular forms, one can extend this definition as follows:

Definition 4.2.6. If $\mathfrak{U}$ is an affinoid subdomain of $\mathcal{W}^G$ defined over a finite extension $L/\mathbb{Q}_p$ and $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ is the restriction of the universal character to $\mathfrak{U}$, then we define

$$\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L) := \mathcal{A}_{n^{\mathfrak{U}}, v^{\mathfrak{U}}}(\mathcal{O}_p, L),$$

with the action of Δ defined analogously where $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}}, n^{\mathfrak{U}}, v^{\mathfrak{U}}, w^{\mathfrak{U}})$ is a weight tuple.

It follows from [Urb11, Lemma 3.4.6], that there exists a smallest integer $m(\mathfrak{U})$ such that $(\kappa^{\mathfrak{U}}, r^{\mathfrak{U}})$ is $m(\mathfrak{U})$-analytic. Moreover,

$$\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L) = \bigcup_{m \geq m(\mathfrak{U})} \mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m).$$

Lemma 4.2.7. *Let $\mathfrak{U} \subset \mathcal{W}^G$ be an affinoid subdomain defined over L, (κ, r, n, v, w) a weight tuple with $(\kappa, r) \in \mathfrak{U}(\overline{\mathbb{Q}}_p)$, and $m \geq m(\mathfrak{U})$. Then there is a canonical bicontinuous isomorphism*

$$\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m) \cong \mathcal{O}(\mathfrak{U}) \widehat{\otimes}_L \mathcal{A}_{n,v}(\mathcal{O}_p, L, m).$$

In particular, $\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m)$ is a non-trivial $\mathcal{O}(\mathfrak{U})$-ONable Banach space and for $m \geq m(\mathfrak{U})$, the inclusion map $\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m) \subset \mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, m+1)$ is completely continuous.

Proof. This follows from [Urb11, Section 3.4.4]. Specifically, it follows from Lemma 3.4.9, Corollary 3.4.10 and Remark 3.4.12 of *loc. cit.* □

4.2.8 Overconvergent quaternionic modular forms

Definition 4.2.9. Let (κ, r, n, v, w) be a weight tuple with $(\kappa, r) \in \mathcal{W}^G(L)$ and U be an open compact subgroup of $G_D(\mathbb{A}_f)$, such that its image under the projection $U \to D_p^\times$ lies in I_m for some $m \geq 1$ and $t \in \mathbb{Z}_{\geq 0}^{\Sigma_p}$ is such that $t + m \geq m(\kappa)$. The space of overconvergent quaternionic modular forms of weight κ, level U and radius of overconvergence p^{-t}, denoted $S_{\kappa,r}^{D,\dagger}(U, t)$ is the space of functions

$$f : G_D(\mathbb{A}_f) \longrightarrow \mathcal{A}_{n,v}(\mathcal{O}_p, L, t)$$

such that

(a) For $d \in G_D(\mathbb{Q})$, we have $f(dg) = f(g)$ for all $g \in G_D(\mathbb{A})$.

(b) For $\gamma \in U$ we have $f(g) = f(g\gamma^{-1}) \cdot \gamma_p$ for all $g \in G_D(\mathbb{A})$, where γ_p is the p-part of γ.

If $\mathfrak{U} \subset \mathcal{W}^G$ is an affinoid subdomain defined over L and $t \geq m(\mathfrak{U})$, then define $S_{\mathfrak{U}}^{D,\dagger}(U, t)$ to be the space of functions

$$f : G_D(\mathbb{A}_f) \longrightarrow \mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L, t)$$

satisfying $(a), (b)$ above. Lastly, taking the limit over t we obtain Fréchet spaces $S_{\kappa,r}^{D,\dagger}(U)$.

We now have the following useful result which describes how the radius of overconvergence and the level are linked.

Proposition 4.2.10. *Let $U = U_*(\mathfrak{n})$ for $* \in \{\emptyset, 0, 1\}$ with $p \nmid \mathfrak{n}$ and let $(\kappa, r) \in \mathcal{W}^G$. If we take s_1, s_2 such that $s_1 + s_2 + t \geq m(\kappa)$, then we have a canonical Hecke equivariant isomorphism*

$$S^{D,\dagger}_{\kappa,r}(U \cap U_0(\pi^{s_2}), t + s_1) \cong S^{D,\dagger}_{\kappa,r}(U \cap U_0(\pi^{s_1+s_2}), t).$$

Proof. See [Buz07, Proposition 11.1]. □

Remark 4.2.11. Note that in this way we can view classical forms of level $U_0(\pi^s)$ and non-trivial character at p as 'part of' our eigenvariety of level $U_0(\pi)$.

4.3 Hecke operators and the Control Theorem

Following Section 12 of [Buz07], we define the Hecke operators on these spaces.

Definition 4.3.1. For $U' = U \cap U_i(\pi^s)$ with $U = U_*(\mathfrak{n})$ with $\mathfrak{n}$ coprime to π, we call $\mathfrak{n}$ the *tame level* and π^s the *wild level.*

Notation 4.3.2. If v is a finite place of F, such that D_{v} is split, then let $\eta_{\mathrm{v}} \in D_f^\times$ be the element which is the identity at all places different from v and at v it is the matrix $\left(\begin{smallmatrix} \pi_{\mathrm{v}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for π_{v} a uniformizer of F_{v}. In order to ease notation later on, when $\mathrm{v}|p$ we choose the same uniformizers as we had before.

Definition 4.3.3. Let U have tame level $\mathfrak{n}$ and wild level π^s. For each v as above, we define the Hecke operators T_{v} as the double coset operators given by $[U\eta_{\mathrm{v}}U]$. Moreover, if v is coprime to level, then we can regard π_v as an element of the centre of $D_f^\times$ and we denote by S_{v} the operator $[U\pi_{\mathrm{v}}U]$. Lastly, for each $\mathfrak{p} \in \Sigma_p$ let $U_{\mathfrak{p}}$ denote the operator $T_{\mathfrak{p}}$ and let $U_p = \prod_{\mathfrak{p} \in \Sigma_p} U_{\mathfrak{p}}$. We denote by $\boldsymbol{T} = \boldsymbol{T}^D(U)$, the Hecke algebra generated by the operators[2] $T_{\mathfrak{q}}, U_{\mathfrak{p}}$, where $\mathfrak{q} \nmid \mathfrak{n}\mathfrak{d}$ with $\mathfrak{d} = \mathrm{Disc}(D)$ and $\mathfrak{p} \in \Sigma_p$.

We now want to show that the overconvergent quaternionic modular forms of small slope are classical. To do this we will follow the proof of the case $F = \mathbb{Q}$ in [Buz04, Section 7]. We begin with some preliminaries.

Lemma 4.3.4. *The U_p operator acting on $S^{D,\dagger}_{\mathfrak{U}}(U \cap U_0(\pi^s), t)$ for $s + t \geq m(\mathfrak{U})$ is compact. In particular, this holds for the spaces $S^{D,\dagger}_{\kappa,r}(U \cap U_0(\pi^s), t)$ for $s + t \geq m(\kappa, r)$.*

Proof. See [Buz07, Lemma 12.2] or [Urb11, Lemma 3.2.8].

□

Proposition 4.3.5. *Let $f \in S^{D,\dagger}_{\kappa,r}(U \cap U_0(\pi^s), t)$ with $s + t \geq m(\kappa, r)$. If $U_p f = \lambda f$ for some non-zero λ, then $f \in S^{D,\dagger}_{\kappa,r}(U \cap U_0(\pi^s), t')$, for any $t' \leq t$ such that $t' + s \geq m(\kappa)$.*

[2]Note that these operators are independent of choice of uniformizer for v not dividing p.

Proof. This follows easily by noting that $U_p f \in S_{\kappa,r}^{D,\dagger}(U \cap U_0(\pi^s), t-1)$ and then using Proposition 4.2.10.

$\square$

Definition 4.3.6. Let (κ, r) be an algebraic weight and let

$$\kappa_i = (k_1, \ldots, k_{i-1}, 2-k_i, k_{i+1}, \ldots k_g).$$

Note that if $\kappa = 2w - r$ then $\kappa_i = 2w' - r$ where $w'_j = w_j$ for $j \neq i$ and $w'_i = v_i$. For each $i \in \{1, \ldots, g\}$ corresponding to a place in Σ_∞, we define a map

$$\Theta_i : S_{\kappa,r}^{D,\dagger}(U, 0) \longrightarrow S_{\kappa_i,r}^{D,\dagger}(U, 0)$$

by setting

$$\Theta_i(f)(h) = \frac{\partial^{k_i-1} f(h)}{\partial z_i^{k_i-1}}$$

for $h \in G_D(\mathbb{A}_f)$.

Note that $f(h) \in \mathcal{A}_{n,v}(\mathcal{O}_p, L, 0)$ so it can be written as a converging power series in variables $(z_1, \ldots, z_g)$, so $\frac{\partial^{k_i-1} f(h)}{\partial z_i^{k_i-1}}$ makes sense. Moreover, one needs to check that Θ_i is actually well-defined, but this follows at once from the simple check that for any $\gamma \in I$ we have $\Theta_i(f)|_\gamma = \Theta_i(f|_\gamma)$.

Theorem 4.3.7 (Control Theorem). *Let $U' = U_*(\mathfrak{n})$ with $(\mathfrak{n}, \pi) = 1$ and $U = U' \cap U_1(\pi^s)$ for $s \geq 1$ and let (k, r) be a classical weight. Let $f \in S_{k,r}^{D,\dagger}(U, t)$ be an eigenform for each $U_{\mathfrak{p}_i}$ with eigenvalue $\alpha_{\mathfrak{p}_i}$. If for each $\mathfrak{p}_i | p$ we have*

$$\mathrm{val}_p(\alpha_{\mathfrak{p}_i}) < \frac{v_{\mathfrak{p}_i}(k, r) + \min_{j \in \Sigma_{\mathfrak{p}_i}} \{k_j - 1\}}{e_{\mathfrak{p}_i}},$$

where $e_{\mathfrak{p}_i}$ is the ramification degree, then $f \in S_{k,r}^D(U)$ (in other words, f is classical).

Proof. We will only sketch the proof, but the full details[3] can be found in [Yam07, Theorem 2.3]. First note that if $\Theta_i(f) = 0$ for all i then f must in fact be classical. The task is now to give a criterion for f to be in this kernel based only on the slope of f. Now, let $U_{\mathfrak{p}_i}^0 = \pi_{\mathfrak{p}_i}^{-v_{\mathfrak{p}_i}(k,r)} U_{\mathfrak{p}_i}$ which has operator norm ≤ 1. Then any eigenform of $U_{\mathfrak{p}_i}^0$ with negative slope must in fact be zero. Now Θ_i sends $U_{\mathfrak{p}_i}^0$-eigenforms of slope h to $U_{\mathfrak{p}_i}^0$-eigenforms of slope $h - \frac{\min_{j \in \Sigma_{\mathfrak{p}_i}} \{k_j - 1\}}{e_{\mathfrak{p}_i}}$, from which one can deduce the result.

$\square$

[3] Up to normalization

Using the above and the Eigenmachine we can construct the eigenvariety associated to overconvergent quaternionic modular forms for D/F.

Theorem 4.3.8. *Let* $U = U' \cap U_0(\pi)$ *with* U' *having level* $\mathfrak{n}$ *coprime to* π. *Let* $\mathscr{Z}$ *be the spectral variety defined as usual and* $\boldsymbol{T} = \boldsymbol{T}^D(U)$ *as defined in 4.3.3. Lastly, let* $S^{D,\dagger}(U)$ *be the coherent sheaf given the nuclear Frechet spaces* $S^{D,\dagger}_{\mathfrak{U}}(U)$ *where* $\mathfrak{U}$ *is an affinoid with a morphism* $\mathfrak{U} \to \mathcal{W}^G$. *Associated to the eigendata of* $(\mathcal{W}^G, S^{D,\dagger}(U), \boldsymbol{T}, U_p)$ *we have an eigenvariety* $\mathscr{X}_D(U)$ *which is equidimensional of dimension* $g+1$ *and satisfies the conditions of Theorem 2.4.5.*

Proof. The existence of such an eigenvariety and the fact that it is equidimensional follows from [Buz07, Section 13]. The fact that is it is equidimensional of dimension $g+1$ is due to the weight space that we have used. ☐

Chapter 5

Overconvergent cohomology groups

In this section we will very briefly recall the construction of the eigenvarieties associated to overconvergent cohomology groups. The full details of this construction can be found in [Han].

5.1 Classical cohomology groups

Let $Y_D(U)$ be a Shimura variety as in Section 1.3 associated to a quaternion algebra D (not necessarily totally definite) and a sufficiently small level U. We define *local systems* on $Y_D(U)$.

Definition 5.1.1. (a) If N is a right U-module, denote by $\mathcal{L}(N)$ the sheaf of locally constant sections of the cover

$$\widetilde{N} := G_D(\mathbb{Q})\backslash(G_D(\mathbb{A}) \times N)/UC_\infty^+ \longrightarrow Y_D(U),$$

with left action of $G_D(\mathbb{Q})$ and right action of UC_∞^+ on $G_D(\mathbb{A}) \times N$ given by

$$\gamma \cdot (g, n) \cdot uc = (\gamma guc, nu)$$

for $\gamma \in G_D(\mathbb{Q})$, $g \in G_D(\mathbb{A})$, $n \in N$, $u \in U$ and $c \in C_\infty^+$ (see 1.2.2 for the definition of C_∞^+).

(b) If M is a $\mathbb{Q}$ vector space with a right action by $G_D(\mathbb{Q})$, we again denote by $\mathcal{L}(M)$ the sheaf of locally constant sections of the cover

$$\widetilde{M} := G_D(\mathbb{Q})\backslash(G_D(\mathbb{A}) \times M)/UC_\infty^+ \longrightarrow Y_D(U),$$

with left action by $G_D(\mathbb{Q})$ and right action by UC_∞^+ on $G_D(\mathbb{A}) \times M$ given by $\gamma \cdot (g, n) \cdot uc = (\gamma guc, n\gamma)$ for $\gamma \in G_D(\mathbb{Q})$, $g \in G_D(\mathbb{A})$, $n \in N$, $u \in U$ and $c \in C_\infty^+$.

In both cases the local systems will be trivial if $Z_U = Z_{G_D(\mathbb{Q})} \cap U$ does not act trivially on N or M, where Z_{G_D} is the center of G_D.

Remark 5.1.2. Note that it is possible to have a module M, which is simultaneously a right U and $G_D(\mathbb{Q})$ module. In particular, if we are in the case where M is a $G_D(\mathbb{Q}_p)$-module, and both U and $G_D(\mathbb{Q})$ act through this action, then in this case it is easy to see that both the local systems defined above will be isomorphic.

Now choose a finite resolution $F_\bullet(t_i) \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ by $\mathbb{Z}[\Gamma^i(U)]$ modules of finite rank (here t_i and $\Gamma^i(U)$ are as in 1.3.2 and 1.3.4). This is called a Borel-Serre resolution, since its existence relies on taking the Borel-Serre compactification of a certain manifold (see [Urb11, Lemma 4.2.2]).

Definition 5.1.3. Let N be a left U-module and set

$$C^\bullet(Y_D(U), N) = \bigoplus_i \mathrm{Hom}_{\mathbb{Z}[\Gamma_i]}(F_\bullet(t_i), N).$$

If we take this complex and compute its cohomology groups, we get $\mathrm{H}^\bullet(Y_D(U), \mathcal{L}(N))$.

Remark 5.1.4. One can define an action of the Hecke algebra on these cohomology groups or directly on the complex. For the relevant definitions see [BS13, Section 1.5.2] or [Urb11, Section 4.2.6].

We now need to define some local systems to work with.

Definition 5.1.5. Let $V_{n,v}(L)$ be as in Definition 4.1.2. This then gives us a local system whose cohomology groups $\mathrm{H}^\bullet(Y_D(U), \mathcal{L}(V_{n,v}(L)^\vee))$ are *classical cohomology groups.* Here $V_{n,v}(L)^\vee$ denotes the L dual of $V_{n,v}(L)$.

Remark 5.1.6. In the above definition one usually takes $L = \mathbb{C}$, but since we are interested in constructing eigenvarieties, we will work with L being a 'sufficiently large' finite complete extension of $\mathbb{Q}_p$. But note that if we tensor the resulting spaces by $\mathbb{C}$, then we will end up with the same (classical) cohomology groups.

We now have the following important result relating the spaces of modular forms to the classical cohomology groups.

Theorem 5.1.7. *(Eichler-Shimura) Let F be a totally real number field, with $[F : \mathbb{Q}] = g$, D/F a quaternion algebra and $q = |\Sigma_D|$. Then,*

- *Let $D = M_2(F)$. If $|F : \mathbb{Q}|$ is odd or $|F : \mathbb{Q}|$ is even and (k, r) an algebraic weight not of the form $(2, r)$ we have*

$$\bigoplus_{J \subset \Sigma_F} S_{k,r,J}(U) \xrightarrow{\sim} \mathrm{H}^g_{cusp}(Y(U), \mathcal{L}(V_{n,v}(L)^\vee)).$$

- *Let D a division quaternion algebra. If $|\Sigma_D|$ is odd or $|\Sigma_D|$ is even and (k, r) an algebraic weight not of the form $(2, r)$ we have*

$$\bigoplus_{J \subset \Sigma_D} S^D_{k,r,J}(U) \xrightarrow{\sim} \mathrm{H}^q(Y_D(U), \mathcal{L}(V_{n,v}(L)^\vee)).$$

Proof. This theorem is usually stated for $\mathbb{C}$ coefficients, but there is no problem replacing this with L as above. For $D \neq M_2(F)$, see [Hid88, Theorem 6.2] and [Hid94, Proposition 3.1]. For $M_2(F)$ see [BS13, Proposition 6.4].

□

Here $\mathrm{H}^g_{cusp}(Y(U), \mathcal{L}(V_{n,v}(L)^\vee))$ is the cuspidal cohomology which can be shown to be a direct summand of $\mathrm{H}^g(Y(U), \mathcal{L}(V_{n,v}(L)^\vee))$ (cf. [Han, Section 3.2]).

Remark 5.1.8. For weights of the form $(2, r)$ and $|\Sigma_D|$ even one needs to add a summand to the right hand side of the above equations. For D a totally definite quaternion algebra, this summand is exactly the space $\mathrm{Inv}(U)$ from Definition 1.2.9 (e).

5.2 Overconvergent cohomology groups

Definition 5.2.1. Let $\lambda \in \mathrm{Hom}_{cts}(T(\mathbb{Z}_p), L^\times)$ be a weight coming from $\mathcal{W}^G$ (cf. 3.4.15). Let $\mathscr{D}_\lambda(L)$ be the L-linear L-dual of $\mathcal{A}_\lambda(\mathcal{O}_p, L)$, i.e.

$$\mathscr{D}_\lambda(L) = \mathrm{Hom}^{cts}_L(\mathcal{A}_\lambda(\mathcal{O}_p, L), L).$$

Similarly, define $\mathcal{D}_{\lambda,m}(L) = \mathrm{Hom}^{cts}_L(\mathcal{A}_\lambda(\mathcal{O}_p, L, m), L)$ as the continuous L-dual of $\mathcal{A}_\lambda(\mathcal{O}_p, L, m)$, and we give $\mathcal{D}_{\lambda,m}$ a dual continuous left action of Δ denoted by $\star$. We note that the action is such that for $d \in T^{++}$, $d\star$ factors through $\mathcal{D}_{\lambda,n+1} \hookrightarrow \mathcal{D}_{\lambda,m}$. Lastly, for $\mathfrak{U} \subset \mathcal{W}^G$ an affinoid subdomain, we let $\mathcal{D}_{\mathfrak{U},m} = \mathrm{Hom}^{cts}_{\mathcal{O}(\mathfrak{U})}(\mathcal{A}_{\mathfrak{U}}(\mathcal{O}_p, L), \mathcal{O}(\mathfrak{U}))$ and $\mathscr{D}_{\mathfrak{U}} = \mathrm{Hom}^{cts}_{\mathcal{O}(\mathfrak{U})}(\mathcal{A}_{\mathfrak{U}}(L), \mathcal{O}(\mathfrak{U}))$.

Proposition 5.2.2. *The assignment $\mathfrak{U} \mapsto \mathscr{D}_{\mathfrak{U}}$ defines a Fréchet sheaf on $\mathcal{W}^G$.*

Proof. See [Han, Section 2]. □

Now, in order to define our eigenvarieties, we will be interested in the following cohomology groups.

Definition 5.2.3. Let U be as above, $\lambda_{\kappa,r} \in \mathcal{W}^G$ and $\mathfrak{U}$ an open affinoid of $\mathcal{W}^G$. Then $H^\bullet(Y_D(U), \mathcal{L}(\mathscr{D}_{\lambda_{\kappa,r}}))$ are the ***overconvergent cohomology groups*** attached to D and $H^\bullet(Y_D(U), \mathcal{L}(\mathscr{D}_{\mathfrak{U}}))$ is a family of overconvergent cohomology groups. We will sometimes drop the $\mathcal{L}$ to ease notation.

Remark 5.2.4. One can define an action of U_p on these cohomology groups and show that it is a compact operator, from which we can then define the slope decompositions of these cohomology groups. See [Urb11, Section 1.2.5] or [Han, Section 2.1].

In this setting we again have a control theorem.

Theorem 5.2.5. *(Ash-Stevens, Urban) Fix an arithmetic weight $\lambda_{k,r}$ and let U be a compact subgroup of $G_D(\mathbb{A}_f)$ of wild level π^m and sufficiently small. If $h < \min_i k_i - 1$ and $m \geq m(\lambda_\kappa)$, then we have a natural isomorphism of Hecke modules*

$$\mathrm{H}^\bullet(Y_D(U), \mathscr{D}_{\lambda_{k,r}})^{\leq h} \cong \mathrm{H}^\bullet(Y_D(U), V_{\lambda_{k,r}}(L)^\vee)^{\leq h}.$$

Proof. See [Urb11, Proposition 4.3.10] or [Han, Theorem 3.2.5].

□

Now in order to define the eigenvariety associated to these overconvergent cohomology groups, we first need to define the generalized eigendata. Most of this will be the same as in the previous sections, but we need to define the spectral variety and the coherent sheaf. This can be found in [Han, Section 4.3], but we briefly recall the main ideas here. For $m \geq m(\mathfrak{U})$ we have a well-defined action of U_p on $C^\bullet(Y_D(U), \mathscr{D}_{\mathfrak{U},m})$, so we can attach a Fredholm series

$$f_{\mathfrak{U}} = \det(1 - U_p X \mid C^\bullet(Y_D(U), \mathscr{D}_{\mathfrak{U}}))$$

to this action. By Tate's acyclicity theorem, we can then find $f \in \mathscr{O}(\mathcal{W}^G)\{\{X\}\}$ with $f\mid_{\mathfrak{U}}(X) = f_{\mathfrak{U}}(X)$. With this we define the spectral variety in this setting to be given by the Fredholm hypersurface $\mathscr{Z}_f = \mathscr{Z}(f)$.

We now have the following result of Hansen, which allows us to define a complex of coherent analytic sheaves on $\mathscr{Z}_f$.

Proposition 5.2.6 (Hansen). *We have a unique complex $\mathcal{C}^\bullet$ of coherent analytic sheaves on $\mathscr{Z}_f$, such that for any slope-adapted affinoid $\mathscr{Z}_{\mathfrak{U},h}$, we have*

$$\mathcal{C}^\bullet(\mathscr{Z}_{\mathfrak{U},h}) \cong C^\bullet(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h}.$$

Proof. See [Han, Proposition 4.3.1]. □

If we then take the cohomology of this sheaf we get a graded sheaf $\mathscr{N}^*_{D,U}$ on $\mathscr{Z}_f$ with a canonical isomorphism

$$\mathscr{N}^*_D(\mathscr{Z}_{\mathfrak{U},h}) \cong \mathrm{H}^*(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h} := \oplus_i \mathrm{H}^i(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h},$$

and by [Han, Proposition 3.15] the natural maps

$$\boldsymbol{T} \longrightarrow \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{\mathfrak{U},h})}(\mathrm{H}^*(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h})$$

glue to give a algebra homomorphism $\psi : \boldsymbol{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{Z}}}(\mathscr{N}^\bullet_{D,U})$ which preserves the grading. Here $\boldsymbol{T} = \boldsymbol{T}^D(U)$ is the Hecke algebra as in Definition 4.3.3.

Proposition 5.2.7. *Let $\mathfrak{U}'$ be an affinoid subdomain of $\mathfrak{U}$, then there are canonical isomorphisms*

$$C^\bullet(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h} \otimes_{\mathscr{O}(\mathfrak{U})} \mathscr{O}(\mathfrak{U}') \cong C^\bullet(Y_D(U), \mathscr{D}_{\mathfrak{U}'})^{\leq h}$$

and

$$\mathrm{H}^*(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h} \otimes_{\mathscr{O}(\mathfrak{U})} \mathscr{O}(\mathfrak{U}') \cong \mathrm{H}^*(Y_D(U), \mathscr{D}_{\mathfrak{U}'})^{\leq h}.$$

Proof. This is [Han, 3.1.5]. □

Now we note that a point $\lambda \in \mathfrak{U}$ is not an affinoid subdomain, so we cannot apply the above result to recover $\mathrm{H}^*(Y_D(U), \mathscr{D}_\lambda)^{\leq h}$. In order to recover this space we need the following stronger result.

Theorem 5.2.8 (Hansen). *Let $\mathfrak{V}$ be a rigid Zariski closed subspace of $\mathfrak{U}$ and define $\mathscr{D}_{\mathfrak{V}} = \mathscr{D}_{\mathfrak{U}} \otimes_{\mathscr{O}(\mathfrak{U})} \mathscr{O}(\mathfrak{V})$. Then there is a convergent second quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathscr{O}(\mathfrak{U})}(\mathrm{H}^j(Y_D(U), \mathscr{D}_{\mathfrak{U}})^{\leq h}, \mathscr{O}(\mathfrak{V})) \implies \mathrm{H}^{i+j}(Y_D(U), \mathscr{D}_{\mathfrak{V}})^{\leq h}.$$

Proof. This is part of [Han, Theorem 3.3]. □

Now we will be interested in the eigenvarieties associated to the following generalized eigendata

$$\mathfrak{H}_D = (\mathscr{W}^G, \mathscr{Z}_f, \mathscr{N}^*_{D,U}, \boldsymbol{T}, \psi),$$

where everything is defined as in the previous sections apart from $\mathscr{N}^*$ (here D can be $M_2(F)$, in which case we denote this with subscript G).

Notation 5.2.9. We denote by $\mathscr{H}_D(U)$ and $\mathscr{H}_G(U)$ the eigenvarieties associated to $\mathfrak{H}_D$ and $\mathfrak{H}_G$ respectively.

Remark 5.2.10. The complex $C^\bullet$ above is not canonical, since it depends on choice of Borel-Serre complex, which means that $\mathscr{Z}_f$ and the coherent sheaf $\mathcal{C}^\bullet$ above are not canonical either, but it turns out that the cohomology sheaves $\mathscr{N}^*_{D,U}$ *are* canonical and therefore so are the resulting eigenvarieties.

Chapter 6

p-adic Langlands functoriality over totally real fields

In this chapter we will relate the different eigenvarieties we have defined in the previous chapters. This will give us an overconvergent Jacquet-Langlands correspondence for Hilbert modular forms which interpolates the classical correspondence. Our results are a natural generalization of Chenevier's results ([Che05]) to Hilbert modular form setting, but we note that in the case of number fields of even degree our corresponding results are stronger than those in *loc.cit.*, since in these cases we obtain isomorphisms between the relevant eigenvarieties and rather than closed immersions.

6.1 The totally definite case

In this section we will prove the following:

Theorem 6.1.1. *Let D be a totally definite quaternion algebra with $\operatorname{Disc} D = \mathfrak{d}$ and $\mathfrak{n}$ an ideal with $(\mathfrak{n}, \mathfrak{d}) = 1$. Let $\mathscr{X}_G := \mathscr{X}_G(U_1(\mathfrak{n}\mathfrak{d}))$ and $\mathscr{X}_D := \mathscr{X}_D(U_1(\mathfrak{n}))$ be as in Theorems 3.4.29 and 4.3.8 respectively, with $U_1(\mathfrak{n}\mathfrak{d})$ a level whose associated moduli problem is representable[1] and let p be unramified. Then these eigenvarieties are reduced and the classical Jacquet-Langlands correspondence can be interpolated to obtain a closed immersion $\mathscr{X}_D \hookrightarrow \mathscr{X}_G$ satisfying the properties of Theorem 2.5.5.*

Corollary 6.1.2. *If $g = [F : \mathbb{Q}]$ is even, then taking D (totally definite) with $\mathfrak{d} = 1$, the closed immersion given by Theorem 6.1.1 becomes an isomorphism.*

We will derive Theorem 6.1.1 from Theorem 2.5.5 (the Interpolation theorem). To this end, we need to exhibit a very Zariski dense set $X \subset \mathcal{W}^G$ on which we can put

[1] Meaning that the moduli problem of HBAV with a $\mu_{\mathfrak{n}\mathfrak{d}}$-level structure is representable.

classical structures for both sets of eigenvariety data. The set of all classical weights (see Definition 3.4.9) is such a candidate. The fact that it is a very Zariski dense subset of $\mathcal{W}^G$ is a well-known fact but we include its proof for the sake of completeness. This requires the following lemma.

Lemma 6.1.3. *If W is a non-empty rigid space. Then W is irreducible if and only if the only analytic subset $Z \subset W$ which set-theoretically contains a non-empty admissible open of W is $Z = W$.*

Proof. See [Con99, Lemma 2.2.3]. $\square$

Proposition 6.1.4. *Let X be the set of classical weights, then X is very Zariski dense in $\mathcal{W}^G$.*

Proof. This is a simple generalization of [Che04, Proposition 6.2.7] or [Ram09, Lemma 4.1]. By 3.4.3 we have

$$\mathcal{W}^G \cong H^\vee \times B(1,1)^{g+1} \cong \bigsqcup_{\chi} \mathcal{W}_\chi,$$

where we index over the elements of $H^\vee$. Let $\kappa_{\psi\tau^a}$ be a classical weight with $\kappa = 2w - r$, ψ and τ^a as in Definition 3.4.9. Then under the above isomorphism

$$\kappa_{\psi\tau^a} \mapsto \left(\widehat{\kappa}_{\psi\tau^a}, \left[\prod_j^g (1+p)^{w_j} \right], (1+p)^r \right),$$

where $\widehat{\kappa}_{\psi\tau^a}$ denotes the restriction to $H^\vee$ (note that $\kappa_{\psi\tau^a} \in \mathcal{W}^G(E)$, with $E = \mathbb{Q}_p[\psi]$). Assume that $\kappa_{\psi\tau^a} \in \mathcal{W}_\chi$ for some χ and take any irreducible admissible affinoid open $V \subset \mathcal{W}^G$ that contains $\kappa_{\psi\tau^a}$. Then $V \subset \mathcal{W}_\chi$ and moreover, since $V(E)$ is open, there exists $\boldsymbol{s} = (s, s') \in \mathbb{Q}_{>0}^{\Sigma_\infty} \times \mathbb{Q}_{>0}$ such that the closed ball of radius $\boldsymbol{s}$ around $\kappa_{\psi\tau^a}$ is contained in V, i.e.,

$$B[\kappa_{\psi\tau^a}, \boldsymbol{s}] := \prod_j^g B[w_j, s_j] \times B[r, s'] \subset V.$$

By Lemma 6.1.3, we see that if $B[\kappa_{\psi\tau^a}, \boldsymbol{s}] \cap X$ is Zariski dense in $B[\kappa_{\psi\tau^a}, \boldsymbol{s}]$, then $V(E) \cap X$ is Zariski dense in V, which is what we want to prove. So we are reduced to showing that $B[\kappa_{\psi\tau^a}, \boldsymbol{s}](E) \cap X$ is Zariski dense in $B[\kappa_{\psi\tau^a}, \boldsymbol{s}]$. To see this, let $\kappa, \kappa' \in X$, then

$$
\begin{aligned}
|\kappa - \kappa'| &= \max_i \{|(1+p)^{w_i} - (1+p)^{w_i'}|_p, |(1+p)^r - (1+p)^{r'}|_p\} \\
&= \max_i \{|(1+p)^{w_i - w_i'} - 1|_p, |(1+p)^{r-r'} - 1|_p\}.
\end{aligned}
$$

So taking $\kappa'_{\psi\tau^b} \in X$, with $\kappa' = 2w' - r'$ is such that:

- for N large enough, $w_i \equiv w_i' \mod (p-1)p^N$, $r \equiv r' \mod (p-1)p^N$;
- $\kappa_{\psi\tau^a}$ and $\kappa'_{\psi\tau^b}$ lie in the same component of the weight space.

Then we can easily see that $B[\kappa_{\psi\tau^a}, \boldsymbol{s}]$ contains infinitely many elements of X, hence we get the result. $\square$

Definition 6.1.5. Let $\mathscr{Z} \subset \mathcal{W}^G \times \mathbb{A}^1$ be the spectral variety defined by $\mathrm{Fred}_{\mathscr{M}}(U_p)$ for $\mathscr{M}$ (as usual) the coherent sheaf on $\mathcal{W}^G$ of overconvergent Hilbert modular forms on D or $M_2(F)$ with a classical structure $\mathscr{M}^{cl}$. We call a point $z \in \mathscr{Z}$ classical if its projection to $\mathcal{W}^G$ is a classical weight and if $\det(1 - TU_p|_{\mathscr{M}^{cl}})$ vanishes at z. We denote these points by $\mathscr{Z}^{cl}$.

Remark 6.1.6. Note that if $\mathscr{M}$ is given by the 'spreading out' of the spaces of overconvergent Hilbert modular forms and if we write $z = ((\kappa_z, r_z), \alpha) \in \mathcal{W}^G \times \mathbb{A}^1$ then z is classical if (κ_z, r_z) is a classical weight and there exists a classical Hilbert modular form with U_p eigenvalue α^{-1}.

Proposition 6.1.7. *The subset $\mathscr{Z}^{cl}$ of $\mathscr{Z}$ is a very Zariski dense subset.*

Proof. This follows from the proof of [Che05, Proposition 3.5]. The basic idea is to use the fact that X and X_h are very Zariski dense, together with the fact that the admissible cover of $\mathscr{Z}$ as given by [Buz07, Section 4] is finite flat over its projection to weight space. $\square$

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ with $(\mathfrak{n}, \mathfrak{d}) = 1$ and $\pi \nmid \mathfrak{n}\mathfrak{d}$, where $\mathfrak{d} = \mathrm{Disc}(D)$. Let $U^D = U_1(\mathfrak{n}\pi)$ and set $\boldsymbol{T}^D(U^D)$ to be the Hecke algebra.[2] Let U^G be the corresponding level structure when one takes $D = M_2(F)$, which gives the level structure in the Hilbert modular form case. By fixing a splitting at places away from $\mathfrak{d}$, we let $\boldsymbol{T}^D$ act on the spaces of Hilbert modular forms. Therefore, throughout this section we denote $\boldsymbol{T}^D$ simply by $\boldsymbol{T}$.

Remark 6.1.8. We note that, for g even, we can pick the quaternion algebra D to be totally definite with $\mathfrak{d} = 1$. Now, by fixing a splitting we can identify U^D and U^G, which

[2]Note that the Hecke algebra consists of all Hecke operators away from $\mathfrak{d}$.

we will simply denote by U. In this case the classical Jacquet-Langlands correspondence (Theorem 1.4.5) gives an isomorphism of Hecke modules

$$S^D_{k,r}(U) \xrightarrow{\sim} S_{k,r}(U).$$

However, for g odd, since D is totally definite, we must have $\mathfrak{d} \neq 1$. In this case, Theorem 1.4.5 gives an isomorphism of Hecke modules

$$S^D_{k,r}(U^D) \xrightarrow{\sim} S^{\mathfrak{d}\text{-new}}_{k,r}(U^G(\mathfrak{d})) \hookrightarrow S_{k,r}(U^G(\mathfrak{d})),$$

where $U^G(\mathfrak{d}) = U^G \cap U_1(\mathfrak{d})$.

Theorem 6.1.1 then follows from Theorem 6.1.9 (below) together with Lemma 6.1.12:

Theorem 6.1.9. *Let $\mathscr{X}_G$ and $\mathscr{X}_D$ be the eigenvarieties associated to the eigendata*

$$\mathfrak{D}_1 = (\mathcal{W}^G, S^{G,\dagger}(U^G(\mathfrak{d})), \boldsymbol{T}, U_p)$$

and

$$\mathfrak{D}_2 = (\mathcal{W}^G, S^{D,\dagger}(U^D), \boldsymbol{T}, U_p)$$

respectively and let p be unramified. Then we can interpolate the classical Jacquet-Langlands correspondence and obtain a closed immersion $\iota_D : \mathscr{X}^{red}_D \hookrightarrow \mathscr{X}^{red}_G$.

Proof. We will prove this using Theorem 2.5.5. Let X be the set of classical weights, whose elements we will denote by k. We now define classical structure on X. For each $(k,r) \in X$, let $\mathscr{M}^{cl}_{G,k,r}$ and $\mathscr{M}^{cl}_{D,k,r}$ be the $\boldsymbol{T}$-modules $S_{k,r}(U^G(\mathfrak{d}))$ and $S^D_{k,r}(U^D)$ respectively of classical cusp forms of weight k and level $U^G(\mathfrak{d}), U^D$ respectively. We need to check that this is indeed a classical structure. Pick $h \in \mathbb{R}_{\geq 0}$. Then the set of $(k,r) \in X$ such that $S^{G,\dagger}_{k,r}(U^G(\mathfrak{d}))^{\leq h} \subset \mathscr{M}^{cl}_{G,k,r}$ contains all $(k,r) \in \mathcal{W}^G$, such that

$$h < v_p(k,r) + \min_{i \in \Sigma_\infty} \{k_i - 1\}$$

by Theorem 3.4.28, and hence satisfies the properties of Definition 2.5.3. Recall that the superscript $\leq h$ denotes slope decomposition with respect to U_p.

Similarly, if $(k,r) \in X$ is such that[3]

$$h < v_p(k,r) + \min_{i \in \Sigma_\infty} \{k_i - 1\},$$

[3]Note that we are in the case where p is unramified.

then $S^{D,\dagger}_{k,r}(U^D)^{\leq h} \subset \mathscr{M}^{cl}_{D,k,r}$. It follows that we again have a classical structure. Now, as a consequence of the classical Jacquet-Langlands correspondence we have that

$$\det\left(1 - U_p X_{|\mathscr{M}^{cl}_{D,k,r}}\right) \text{ divides } \det\left(1 - U_p X_{|\mathscr{M}^{cl}_{G,k,r}}\right).$$

Hence we can apply Theorem 2.5.5 to obtain the closed immersion $\iota_D : \mathscr{X}^{red}_D \hookrightarrow \mathscr{X}^{red}_G$. $\square$

Now observe that if g is even, then we can pick D to be totally definite *and* have $\mathfrak{d} = 1$ (i.e. trivial discriminant). Then the classical Jacquet-Langlands correspondence gives that $\mathscr{M}^{cl}_{G,k,r} \cong \mathscr{M}^{cl}_D(k)$ at classical weights, and thus

$$\det(1 - U_p X \mid \mathscr{M}^{cl}_D(k)) = \det(1 - U_p X \mid \mathscr{M}^{cl}_{G,k,r})$$

therefore Corollary 2.5.6 gives us an isomorphism $\mathscr{X}^{red}_D \cong \mathscr{X}^{red}_G$. This proves most of Corollary 6.1.2, it only remains to show that the eigenvarieties are reduced which we will follow from Lemma 2.5.7.

Proposition 6.1.10. *Fix $h \in \mathbb{R}_{\geq} 0$. There is a Zariski dense subset $X' \subset X$ (of $\mathcal{W}^G$) such that for all $k \in X'$, the $\boldsymbol{T}$-module $\mathscr{M}^{cl,\leq h}_{G,k,r}$ is semisimple.*

This result will be consequence of the two lemmas below. We begin by noting that the classical Jacquet-Langlands correspondence gives us that if $\mathscr{M}^{cl,\leq h}_{G,k,r}$ is a semisimple $\boldsymbol{T}$-module, then so is $\mathscr{M}^{cl,\leq h}_{D,k,r}$. To ease notation, we let

$$V^h_{k,r} := \mathscr{M}^{cl,\leq h}_{G,k,r} = S_{k,r}(U^G(\mathfrak{d}))^{\leq h}.$$

Now, since we are working with classical Hilbert modular forms, the action of the Hecke operators can be described by their action on q-expansions. Next we note that the only Hecke operators that might not be semisimple are the $U_{\mathfrak{p}_i}$, for $p\mathcal{O}_F = \prod_i \mathfrak{p}_i$. This is because all the other operators are normal (commute with their adjoints), so they are semisimple (cf. 3.2.7). Hence we must show that for each i, the operators $U_{\mathfrak{p}_i}$ act semisimply on the space of cusp forms of slope-$\leq h$. In fact we shall show that $U_{\mathfrak{p}_i}$ acts semisimply on $V^h_{k',r'}$ for a Zariski dense subset of $X' \subset X$. Lastly, we need to relate slope decomposition of $V^h_{k,r}$ with respect to U_p, to the slope decompositions with respect to the $U_{\mathfrak{p}_i}$. To do this we have the following:

Lemma 6.1.11. *Let S be a Banach space on which we have pairwise commuting operators U_i for $i = 1, \dots, n$, all of which have operator norm ≤ 1 (which means they have positive slopes) and such that $U = \prod_i U_i$ is a compact operator. Then the slopes of the U_i operators acting on the space $S^{\leq h}$ (this is the slope decomposition with respect to U) are all $\leq h$.*

Proof. By definition we have that $S^{\leq h}$ is a finite dimensional subspace of S. Therefore by choosing a basis we can view the U_i operators as matrices. Now since the U_i are pairwise commuting operators, we can simultaneously upper triangularize them (after possibly extending the base field). From this it follows that the eigenvalues of U acting on $S^{\leq h}$ are the product of the eigenvalues of the U_i.

Now since the slopes of an operator are simply the p-adic valuation of its eigenvalues, we have that on $S^{\leq h}$ the slopes of U are the sum of the slopes of the U_i operators and therefore, since they all have positive slopes, it follows that the slopes of the U_i acting on $S^{\leq h}$ are all $\leq h$ as required.

□

After renormalizing our operators, we can apply this to our situation to see that since $U_p = \prod_i U_{\mathfrak{p}_i}$ is compact, then we have a slope decomposition for any h. Moreover, for each h we have that the slope of each $U_{\mathfrak{p}_i}$ acting on $V_{k,r}^h$ is less than or equal to h. With this we can prove the following Lemma:

Lemma 6.1.12. *There is a Zariski dense subset $X' \subset \mathcal{W}^G$, such that for each i, $U_{\mathfrak{p}_i}$ acts semisimply on $V_{k,r}^h$ for $(k,r) \in X'$.*

Proof. Using Lemma 6.1.11 the result is a simple generalization of the classical situation, as is done in [Bel12, Theorem 3.30], or from the proof of [CE98, Theorem 4.2]. But for completeness we prove it here.

First note that we can decompose $V_{k,r}^h$ into its $\mathfrak{p}_i$-new and $\mathfrak{p}_i$-old parts. The action of $U_{\mathfrak{p}_i}$ on $V_{k,r}^{h,\mathfrak{p}_i\text{-new}}$ is normal and hence diagonalizable. With this we are reduced to showing that this operator acts semisimply on $V_{k,r}^{h,\mathfrak{p}_i\text{-old}}$. In order to prove this, it is enough to show that on each generalized $\boldsymbol{T}$-eigenspace of $V_{k,r}^{h,\mathfrak{p}_i\text{-old}}$ it acts semisimply. Each of these spaces will correspond to a newform f of (lower) level not divisible by $\mathfrak{p}_i$. Now, let $a_{\mathfrak{p}_i} = \mathfrak{a}(\mathfrak{p}_i, \mathbf{f}|T_{\mathfrak{p}_i})$ be the $T_{\mathfrak{p}_i}$ eigenvalue of $\mathbf{f}$ and Ψ its nebentypus. Since we are assuming that for each i, we have $(\mathfrak{n}\mathfrak{d}, \mathfrak{p}_i) = 1$, then it follows from Atkin-Lehner Theory that each of these $\mathfrak{p}_i$-old subspaces is 2-dimensional, and generated by $\mathbf{f}$ and $\iota_{\mathfrak{p}_i}(\mathbf{f})$ (see Section 3.2.8). Proposition 3.2.11 then shows that on this subspace the $U_{\mathfrak{p}_i}$ operator has minimal polynomial given by

$$X^2 - a_{\mathfrak{p}_i} X + N_{F/\mathbb{Q}}(\mathfrak{p}_i)^{r+1}\Psi^*(\mathfrak{p}_i).$$

Therefore, since $N_{F/\mathbb{Q}}(\mathfrak{p}_i) = p^{l_{\mathfrak{p}_i}}$ (here $l_{\mathfrak{p}_i}$ is the residue degree), we see that if we pick (k,r), such that $r > (2h - l_{\mathfrak{p}_i})/l_{\mathfrak{p}_i}$, then $h < l_{\mathfrak{p}_i}(r+1)/2$. Therefore, the polynomial must have a unique root α with valuation $\leq h$, from which it follows that on the generalized $\boldsymbol{T}$-eigenspace of $V_{k,r}^{h,\mathfrak{p}_i\text{-old}}$ corresponding to $\mathbf{f}$, we have that $U_{\mathfrak{p}_i}$ acts as the scalar α. Hence it is diagonalizable. This then shows that on $V_{k,r}^h$ the $U_{\mathfrak{p}_i}$ operators act semisimply for (k,r) large enough as required.

□

From this it follows that for any $h \geq 0$, the operators $U_{\mathfrak{p}_i}$ act semisimply on $S_{k,r}(U')^{\leq h}$ for (k,r) in a Zariski dense subset of $\mathcal{W}^G$, proving Proposition 6.1.10. Then by Lemma 2.5.7, we have at once that $\mathscr{X}_D^{red} \cong \mathscr{X}_D$ and $\mathscr{X}_G^{red} \cong \mathscr{X}_G$, which proves Corollary 6.1.2.

Remark 6.1.13. In light of Theorem 6.1.9 and Remark 6.1.8, we see that for g even and D totally definite with $\mathfrak{d} = 1$, we have an isomorphism of eigenvarieties $\iota_D : \mathscr{X}_D^{red} \xrightarrow{\sim} \mathscr{X}_G^{red}$. However, for g odd and D totally definite, the closed immersion $\iota_D : \mathscr{X}_D^{red} \hookrightarrow \mathscr{X}_G^{red}$ is never an isomorphism since $\mathfrak{d} \neq 1$. At best, we can say that its image is the $\mathfrak{d}$-new part of of $\mathscr{X}_G^{red}$ as in the case of modular forms over $\mathbb{Q}$ (cf. [Che05]).

Remark 6.1.14. We note that, for g odd, there are alternative constructions for the eigenvariety $\mathscr{X}_D^{red}$. Let D be the quaternion algebra ramified at all infinite places but one, with $\mathfrak{d} = 1$. Then Brasca [Bra13] constructs an eigenvariety associated to $\mathscr{X}_D^{red}$ from which one can use the above to obtain an isomorphism $\iota_D : \mathscr{X}_D^{red} \xrightarrow{\sim} \mathscr{X}_G^{red}$. His construction combines the theory of Shimura curves with work of Andreatta-Iovita-Pilloni to construct the relevant eigenvarieties.

6.2 General quaternion algebras

Combining the (classical) Eichler-Shimura isomorphism with the results from the previous section, we can now prove the following theorem:

Theorem 6.2.1. *Let D be any quaternion algebra, U a sufficiently small level and let $\mathscr{H}_D(U), \mathscr{H}_G(U \cap U_0(\mathfrak{d}))$ be the eigenvarieties associated to overconvergent cohomology groups as in 5.2.9. Then these eigenvarieties are reduced and there is a closed immersion*

$$\mathscr{H}_D(U)^o \hookrightarrow \mathscr{H}_G(U \cap U_0(\mathfrak{d}))$$

interpolating the classical Jacquet-Langlands correspondence, where $\mathscr{H}_D(U)^o$ is the core as defined in 2.5.8.

Proof. First recall that the above eigenvarieties have been constructed from the generalized eigendata of $\mathfrak{H}_D = (\mathcal{W}^G, \mathscr{Z}_D, \mathscr{N}_D^*, \boldsymbol{T}, \psi_D)$ and $\mathfrak{H}_G = (\mathcal{W}^G, \mathscr{Z}_G, \mathscr{N}_G^*, \boldsymbol{T}, \psi_G)$. Now, in order to apply Theorem 2.5.10 we need (with the notation as in theorem):

(a) A closed immersion $j : \mathcal{W}_1 \hookrightarrow \mathcal{W}_2$.

(b) A very Zariski dense subset $\mathscr{Z}^{cl} \subset \mathscr{Z}_D^o$ with image in $\mathscr{Z}_G$ under the map induced by j and such that for all $t \in \boldsymbol{T}$ and all $z \in \mathscr{Z}^{cl}$

$$\det\left(1 - \psi_D((tU))Y \mid_{\mathscr{N}_{D,z}^{cl}}\right) \text{ divides } \det\left(1 - \psi_G(tU)Y \mid_{\mathscr{N}_{G,z}^{cl}}\right)$$

in $k(z)[Y]$.

In our setting we have $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}^G$ so (a) is satisfied. For part (b), we let $\mathscr{Z}^{cl}$ be we use subset of $\mathscr{Z}_D$ of points $z \in \mathscr{Z}$ whose projection $\omega(z)$ to $\mathcal{W}^G$ is a classical weight not of the form $(2, r)$ (here we are using 3.4.13) and $\det(1 - TU_p|_{\mathscr{N}_{D,z}^{cl}})$ vanishes at z, where $\mathscr{N}_{D,z}^{cl} = \mathrm{H}^*(Y_D(U), \mathcal{L}(V_{\omega(z)})^\vee)$. Now, just as in Proposition 6.1.7, it follows that $\mathscr{Z}^{cl}$ is a very Zariski dense subset of $\mathscr{Z}_D^o$. We only need to check the divisibility of the Fredholm determinants, but this follows at once from combining the Theorem 5.1.7, Theorem 1.4.5 and Proposition 1.4.4. □

Remark 6.2.2. The advantage of working the overconvergent cohomology is that we do not need to worry about the representabiliy of the moduli problems. On the other hand, since we are using Hansen's more general interpolation theorem, we only have a closed immersion from the core of the quaternionic eigenvariety, but we expect that for D a division quaternion algebra, the resulting eigenvarieties are unmixed.

Part III

Slopes of overconvergent Hilbert modular forms

In this final section we will use the overconvergent Jacquet-Langlands to study slopes of Hilbert modular forms. This reduces us to computing the slopes of overconvergent quaternionic modular forms which are more suited to explicit computations. The algorithms used to compute U_p are very much inspired by [Dem05, Jac03, WXZ14].

We will compute slopes in many cases and make conjectures about their structure. Alongside this, we give some theoretical evidence towards our stated conjectures and prove a lower bound for the Newton polygon of U_p action for any arithmetic weight.

Chapter 7

Quaternionic modular forms over real quadratic fields

Throughout this section F will be a real quadratic field (although most of the theoretical results in this section can easily be extended to any totally real field of even degree). In particular, for computational purposes we will work with $\mathbb{Q}(\sqrt{d})$ where $d = 5, 13, 17$ since these are real quadratic fields for which there exists a totally definite quaternion algebra D/F with trivial discriminant and class number one[1]. The fact that we work with a quaternion algebra that has class number one is simplify computations, and one can most certainly work over any number field of even degree (or possibly any degree) by adapting the work of Dembélé-Voight [DV13], but at the cost of increasing the computational complexity.

In order to make this chapter more self-contained, we recall/simply some of the notation introduced before:

Notation 7.0.1. (1) As before, we let $\mathcal{O}_p := \mathcal{O}_F \otimes \mathbb{Z}_p$ for a rational prime p, which we assume is split or inert in $\mathcal{O}_F$. In the split case we write $p = \mathfrak{p}_1\mathfrak{p}_2$, from which it follows that $\mathcal{O}_p \cong \mathcal{O}_{F_{\mathfrak{p}_1}} \oplus \mathcal{O}_{F_{\mathfrak{p}_2}}$. In this case we take p to be a uniformizing element in each factor, (which we can do since p is split). If we need to distinguish the components we will denote the uniformizers by $\pi_\mathfrak{p}$ as usual. Note that in this case $F_{\mathfrak{p}_i} \cong \mathbb{Q}_p$ and $\mathcal{O}_{F_{\mathfrak{p}_i}} \cong \mathbb{Z}_p$.

When p is inert, $\mathcal{O}_p$ is simply the ring of integers of the degree 2 unramified extension of $\mathbb{Q}_p$, and we again let p be the uniformizer. Lastly, throughout this section we denote our level structures $U_i(\mathfrak{n}\pi^s)$ simply by $U_i(\mathfrak{n}p^s)$, where π is as is Notation 4.1.1.

(2) Let $\psi : (\mathcal{O}_p/p^s)^\times \to \mathcal{O}_{\mathbb{C}_p}^\times$ denote a finite continuous character. We also let ψ denote

[1] In fact $d = 2, 5, 13, 17$ are the only such examples, see [KV10].

the induced character on $\mathcal{O}_p^\times$.

(3) Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ which is coprime to p.

(4) Let L be as before, but enlarged to contain the image of ψ (in particular, in the split case this is some totally ramified extension of $\mathbb{Q}_p$ and in the inert case this is a totally ramified extension of F_p).

From Section 4.1 we have that, in this setting, the spaces of overconvergent quaternionic modular forms are defined as follows:

Definition 7.0.2. Let $D_f^\times := G_D(\mathbb{A}_f)$ and $D^\times := G_D(\mathbb{Q})$. The space of overconvergent quaternionic modular forms of weight (κ_ψ, r), level $U = U_0(\mathfrak{n}p^s)$, and radius of overconvergence p^0, denoted by $S^{D,\dagger}_{\kappa_\psi,r}(U_0(\mathfrak{n}p^s), 0)$, is defined as the vector space of functions

$$f : D^\times \backslash D_f^\times \longrightarrow L\langle X, Y\rangle$$

such that $f(dg) = f(g)$ for all $d \in D^\times$ and $f(gu^{-1}) \cdot_{\kappa_\psi} u_p = f(g)$ for all $u \in U_0(\mathfrak{n}p^s)$ and $g \in D_f^\times$. Here the action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)\right) \in \Delta$ on $L\langle X, Y\rangle$ is given by

$$X^l Y^m \cdot_{\kappa_\psi} \gamma = \psi(d) H(\gamma_1, X, l) H(\gamma_2, Y, m),$$

where

$$H(\gamma_i, Z, t) = (a_i d_i - b_i c_i)^{v_i} (c_i Z + d_i)^{n_i} \left(\frac{a_i Z + b_i}{c_i Z + d_i}\right)^t,$$

Δ is as in Notation 4.1.1 and $(\kappa_\psi, r, n, v, w)$ is a weight tuple.

Remark 7.0.3. In order for the space of modular forms of weight $(\kappa, r)\psi$ to be non-trivial, one requires that $\psi(x) = N_{F/\mathbb{Q}}(x)^r$ for all $x \in \mathcal{O}_F^\times$, which we view as embedded in $\mathcal{O}_p^\times$ in the usual way.

Remark 7.0.4. In the split case, it is clear how to write γ as $\left(\left(\begin{smallmatrix} a_{\mathfrak{p}_i} & b_{\mathfrak{p}_i} \\ c_{\mathfrak{p}_i} & d_{\mathfrak{p}_i} \end{smallmatrix}\right)\right)$ by using the completions of $\mathfrak{p}_1, \mathfrak{p}_2$. In the inert case, we simply let the γ_i be the images of γ under the automorphisms of F_p.

Notation 7.0.5. Throughout this section we will always work with overconvergent modular forms with radius of overconvergence $p^0 = 1$, so we will simply denote these spaces by $S^{D,\dagger}_{\kappa_\psi,r}(U_0(\mathfrak{n}p^s))$. This is not a problem, as one can show that the characteristic polynomial of U_p does not depend on this radius.

Since D is totally definite, we have from before that $Y_D(U) = D^\times \backslash D_f^\times / U$ is just a finite number of points, which we called the class number of (D, U). Moreover, since

D has class number one, then $D_f^\times = D^\times \widehat{\mathcal{O}}_D^\times$ and $D^\times \backslash D_f^\times = \mathcal{O}_D^\times \backslash \widehat{\mathcal{O}}_D^\times$. Thus there is a bijection

$$D^\times \backslash D_f^\times / U \longrightarrow \mathcal{O}_D^\times \backslash \widehat{\mathcal{O}}_D^\times / U$$

and we can write $\widehat{\mathcal{O}}_D^\times = \coprod_{i=1}^h \mathcal{O}_D^\times t_i U$ for t_i suitable representatives. In what follows we will use the above decomposition to write an element $x \in D_f^\times$ as du where $d \in D^\times, u \in \widehat{\mathcal{O}}_D^\times$. Moreover, we can use the above bijection to then write $u = d' t_i \gamma$ (for some i) where $d' \in \mathcal{O}_D^\times$ and $\gamma \in U_0(\mathfrak{n}p^s)$. Now, following Dembélé [Dem05], we find the t_i by observing that

$$\mathcal{O}_D^\times \backslash \widehat{\mathcal{O}}_D^\times / U = \mathcal{O}_D^\times \backslash \mathbb{P}^1(\mathcal{O}_F / \mathfrak{n}p^s)$$

where $\mathbb{P}^1(A) := \left\{(a,b) \in A^2 \mid \exists (\alpha, \beta) \in A^2 \text{ such that } \alpha a - \beta b = 1\right\} / A^\times$. We note that

$$\mathbb{P}^1(\mathcal{O}_F / \mathfrak{n}p^s) = \prod_{\mathfrak{q} \mid \mathfrak{n}p^s} \mathbb{P}^1(\mathcal{O}_F / \mathfrak{q}^{e_\mathfrak{q}}).$$

From this we can find the t_i by simply picking a representative

$$(a, b) = (a_\mathfrak{q}, b_\mathfrak{q})_{\mathfrak{q} \mid \mathfrak{n}p^s} \in \mathbb{P}^1(\mathcal{O}_F / \mathfrak{n}p^s)$$

for each $\mathcal{O}_D^\times$-orbit, and then lifting this to the element of $\widehat{\mathcal{O}}_D^\times$ which is 1 at all places not diving $\mathfrak{n}p$ and, at the places dividing the level, we take $(\alpha_\mathfrak{q}, \beta_\mathfrak{q}) \in (\mathcal{O}_F / \mathfrak{q}^{e_\mathfrak{q}})^2$ such that $a_\mathfrak{q} \alpha_\mathfrak{q} - b_\mathfrak{q} \beta_\mathfrak{q} = 1$ and set $(t_i)_\mathfrak{q} = \begin{pmatrix} a_\mathfrak{q} & b_\mathfrak{q} \\ \beta_\mathfrak{q} & \alpha_\mathfrak{q} \end{pmatrix}$.

Lemma 7.0.6. *There is an isomorphism*

$$S_{\kappa_\psi, r}^{D,\dagger}(U) \xrightarrow{\sim} \bigoplus_{i=1}^h L\langle X, Y \rangle^{\Gamma^i(U)} \tag{7.1}$$

given by sending f to $(f(t_i))_i$, where $\Gamma^i(U)$ is as in 1.3.4.

Proof. Let $f \in S_{\kappa_\psi, r}^{D,\dagger}(U)$. For $g \in D_f^\times$ we can decompose it as $g = dt_i\gamma$ for some i, $d \in D^\times$ and $u \in U$. Now the image of g in (some) $L\langle X, Y \rangle$ under f is given by

$$f(g) = f(dt_i\gamma) = f(t_i\gamma) = f(t_i) \cdot u_p.$$

Therefore it is enough to know where the t_i are sent. But note that if $u \in \Gamma^i(U)$, then $\gamma = t_i^{-1} d t_i$ for some $d \in D^\times$ and thus

$$f(t_i) = f(t_i t_i^{-1} d t_i) = f(t_i) \cdot u_p,$$

from which we see that the image must be in $L\langle X, Y \rangle^{\Gamma^i(U)}$.

$\square$

7.1 The U_p operator

We now study the U_p operator acting on these spaces. In particular, we will describe how one can compute it. From now on we let $U = U_0(\mathfrak{n}p^s)$ be sufficiently small.

Notation 7.1.1. In this chapter we use a slightly different convention for weight $[2,2]$ modular forms on D. It is customary to define $S_2(U)$ as a quotient $S(U)/\operatorname{Inv}(U)$, where $\operatorname{Inv}(U)$ is a subspace of forms that factor through the reduced norm map (cf. Definition 1.2.9 (e)). But for our purposes we do ***not*** quotient out by $\operatorname{Inv}(U)$, so in weight $[2,2]$ our definitions are slightly different from the standard ones, in particular, our spaces are slightly larger $(\dim(S(U)) = \dim(S_2(U)) + 1)$.

Let e denote the fundamental unit in $\mathcal{O}_F^\times$ and let (κ_ψ, r) be an arithmetic weight such that $\kappa_\psi(e) = N_{F/\mathbb{Q}}(e)^r$, which means that $\Gamma^i(U)$ acts trivially on $L\langle X, Y\rangle$ (by our sufficiently small assumption). Then from (7.1) we have the following commutative diagram

$$\begin{tikzcd} S^{D,\dagger}_{\kappa_\psi,r}(U) \arrow[r, "\sim"] \arrow[d, "U_\mathfrak{p}"] & \bigoplus_{i=1}^{h} L\langle X, Y\rangle \arrow[d, "\mathfrak{U}_\mathfrak{p}"] \\ S^{D,\dagger}_{\kappa_\psi,r}(U) \arrow[r, "\sim"] & \bigoplus_{i=1}^{h} L\langle X, Y\rangle. \end{tikzcd}$$

Therefore, in order to compute the action of $U_\mathfrak{p}$, it is enough to compute $\mathfrak{U}_\mathfrak{p}$. Now we have the following well-known result:

Proposition 7.1.2. *Each double coset $[U\eta_\mathfrak{p}U]$ (see Notation 4.3.2) can be written as*

$$\coprod_{\alpha \in \mathcal{O}_\mathfrak{p}/\pi_\mathfrak{p}} U \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ \alpha\pi_\mathfrak{p}^{s_\mathfrak{p}} & 1 \end{pmatrix}.$$

From this it follows that the action of $U_\mathfrak{p}$ is given by

$$(f|U_\mathfrak{p})(g) = \sum_{\alpha \in \mathcal{O}_\mathfrak{p}/\pi_\mathfrak{p}} f|_{u_\alpha}(g)$$

for $g \in D_f^\times$, and where $u_\alpha = \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ \alpha\pi_\mathfrak{p}^{s_\mathfrak{p}} & 1 \end{pmatrix}$.

Definition 7.1.3. For each t_i as above define

$$\Theta(i,j) := \{\alpha \in \mathcal{O}_{\mathfrak{p}_i}/\pi_{\mathfrak{p}_i} \mid t_i u_\alpha^{-1} = d t_j \gamma_\alpha, \text{ for some } d \in D^\times, \gamma_\alpha \in U\}$$

and let $T_{i,j} = \sum_{\beta \in \Theta(i,j)} (\gamma_\beta u_\beta)_p$. Here $u_\beta = \left(\begin{smallmatrix} \pi_\mathfrak{p} & 0 \\ \beta\pi_\mathfrak{p}^{s_\mathfrak{p}} & 1 \end{smallmatrix}\right)$.

Proposition 7.1.4. *The matrices $(\gamma_\beta u_\beta)_p$ are in $\left(\begin{smallmatrix} \pi\mathcal{O}_p^\times & \mathcal{O}_p \\ \pi^s\mathcal{O}_p & \mathcal{O}_p^\times \end{smallmatrix}\right)$ where π^s is the wild level.*

Proof. The proof follows *mutatis mutandis* from the proof of [LWX14, Proposition 3.1]. □

Proposition 7.1.5. *The action of $\mathfrak{U}_\mathfrak{p}$ is given by a $h \times h$ block matrix, whose (i,j)-block is given by the (infinite) matrix of the action of $T_{i,j}$ on $L\langle X, Y\rangle$.*

Proof. By Lemma 7.0.6 we have that the action is given by

$$\begin{aligned}
(f|U_\mathfrak{p})(t_i) &= \sum_{\alpha \in \mathcal{O}_{\mathfrak{p}_i}/\pi_{\mathfrak{p}_i}} f|_{u_\alpha}(t_i) \\
&= \sum_{\alpha \in \mathcal{O}_{\mathfrak{p}_i}/\pi_{\mathfrak{p}_i}} f(t_i u_\alpha^{-1}) \cdot (u_\alpha)_p \\
&= \sum_{j=1}^{h} f(t_j) \cdot \left(\sum_{\beta \in \Theta(i,j)} (\gamma_\beta u_\beta)_p \right)
\end{aligned}$$

which gives the result. □

Similarly we can do all of the above for U_p and this gives the matrix $\mathfrak{U}_p$.

Warning 7.1.6. With these definitions, the $U_\mathfrak{p}$ operators that we get will not be normalized as in [Hid88, Section 3]. For this we need to work with $\pi_\mathfrak{p}^{-v_\mathfrak{p}(k,r)} U_\mathfrak{p}$, which we will do later.

We now show how to write down the matrices $T_{i,j}$. For this we use the standard trick of using a generating function to get the entries of the corresponding matrix.

Proposition 7.1.7. *The generating function for the $|_{\kappa_\psi, r}$ action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)\right) = (\gamma_1, \gamma_2) \in M_2(\mathcal{O}_p)$, with (κ_ψ, r) an arithmetic weight, is given by*

$$\psi(d) \cdot \frac{\det(\gamma_1)^{v_1} (c_1 X + d_1)^{n_1+1}}{(c_1 X + d_1 - a_1 X Z - b_1 Z)} \cdot \frac{\det(\gamma_2)^{v_2} (c_2 Y + d_2)^{n_2+1}}{(c_2 Y + d_2 - a_2 Y W - b_2 W)},$$

where $(\kappa_\psi, r, n, v, w)$ is a weight tuple.

Proof. The action of $|_{\kappa_\psi,r}\gamma$ on $L\langle X,Y\rangle$ is given by

$$X^iY^j|\gamma = \psi(d)H(\gamma_1,X,i)H(\gamma_2,Y,j) = \sum_{l,k} a_{l,k}^{(i,j)}X^lY^k$$

where $H(\gamma_i,Z,t) = \det(\gamma_i)^{v_i}(c_iZ+d_i)^{n_i}\left(\frac{a_iZ+b_i}{c_iZ+d_i}\right)^t$. Now consider the formal sum $G(X,Y,W,Z,\gamma) = \sum_{i,j,l,k} a_{l,k}^{(i,j)}X^lY^kZ^iY^j$, then

$$\begin{aligned}
G(X,Y,W,Z,\gamma) &= \sum_{i,j} Z^iW^j \sum_{l,k} a_{l,k}^{(i,j)}X^lY^k \\
&= \sum_{i,j} Z^iW^j\psi(d)H(\gamma_1,X,i)H(\gamma_2,Y,j) \\
&= E\cdot(c_1X+d_1)^{n_1}(c_2Y+d_2)^{n_2}\sum_i Z^i\left(\frac{a_1X+b_1}{c_1X+d_1}\right)^i\sum_j W^j\left(\frac{a_2Y+b_2}{c_2Y+d_2}\right)^j
\end{aligned}$$

where $E=\psi(d)\det(\gamma_1)^{v_1}\det(\gamma_2)^{v_2}$. The result then follows by noting that

$$\sum_i Z^i\left(\frac{a_1X+b_1}{c_1X+d_1}\right)^i = \frac{1}{1-Z\left(\frac{a_1X+b_1}{c_1X+d_1}\right)}$$

and similarly for the last term.

□

From this we get an expression for $a_{l,k}^{(i,j)}$.

Proposition 7.1.8. *With the notation as above, we have that the coefficient of* $X^iY^kZ^jW^l$ *in* $G(X,Y,Z,W,\gamma)$ *is*

$$\psi(d)\det(\gamma_1)^{v_1}\det(\gamma_2)^{v_2}\cdot C_{n_1}(\gamma_1,j,i)\cdot C_{n_2}(\gamma_2,l,k)$$

where

$$C_w\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right),x,y\right) = \sum_{t=0}^{x}\binom{w-y}{t}\binom{y}{x-t}a^{x-t}c^td^{w-x-t}b^{y-x+t}.$$

Proof. The proof of this expression is a simple matter of expanding the power series, which is an un-illuminating computation. Similar results can be found in [Jac03, Appendix A]. □

7.1.9. In order to write down the matrix for U_p we need to choose a basis of $L\langle X, Y\rangle$. The natural choice of basis for this is the one given by X^iY^j for $i, j \in \mathbb{Z}_{\geq 0}$. Now in order to compute the finite approximations to the infinite matrix of U_p, we will need to choose an ordering of this basis, which is the same as choosing a bijection $Bi : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. In what follows we will choose the 'diagonal' ordering given by

$$Bi(a, b) = \frac{(a + b + 1)(a + b)}{2} + b$$

and $Bi^{-1}(m) = \left(m - \frac{t(t+1)}{2}, \frac{t(t+3)}{2} - m\right)$ where $t = \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor$. Lastly, for $Bi^{-1}(m) = (m_1, m_2)$ set $b(m) = m_1 + m_2 = t$. Note that, from the computational point of view, some choices will mean computation of the slopes stabilize quicker, which is why we use this bijection.

Remark 7.1.10. In what follows the choice of Bi will only be for relevant for computational purposes and not theoretical. The only reason we keep track of if in the results in this section is that we wish to work with matrices and not hypermatrices. Therefore, our theoretical results do not depend in an essential way in our choice of ordering.

It then follows from Proposition 7.1.8 that:

Corollary 7.1.11. *Let $x_1, x_2, y_1, y_2 \in \mathbb{Z}_{\geq 0}$ and*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{pmatrix} \pi_i a_i & b_i \\ c_i \pi_i^{s_i} & d_i \end{pmatrix} \right) \in U.$$

Let $x = Bi(x_1, x_2)$ (similarly for y) and let $(\kappa_\psi, r, n, v, w)$ be a weight tuple with $(\kappa, r)\psi$ an arithmetic weight. Then the x, y entry of the matrix representing the $|_{\kappa_\psi, r}\ \gamma$ action on $L\langle X, Y\rangle$ is given by

$$\Omega_{\kappa_\psi, r}(\gamma, x, y) := E \cdot d_1^{n_1} d_2^{n_2} \pi_1^{x_1} \pi_2^{x_2} \frac{a_1^{x_1}}{d_1^{y_1}} \frac{a_2^{x_2}}{d_2^{y_2}} b_1^{y_1 - x_1} b_2^{y_2 - x_2} C_{n_1}(\gamma_1, x_1, y_1) \cdot C_{n_2}(\gamma_2, x_2, y_2)$$

where $E := \psi(d) \det(\gamma_1)^{v_1} \det(\gamma_2)^{v_2}$ and

$$C_{n_i}(\gamma_i, x_i, y_i) = \sum_{t=0}^{x_i} \binom{n_i - y_i}{t} \binom{y_i}{x_i - t} \left(\frac{b_i c_i}{a_i d_i}\right)^t \pi_i^{t(s_i - 1)}$$

for $i \in \{1, 2\}$ corresponding to $\mathfrak{p}_1, \mathfrak{p}_2$.

Corollary 7.1.12. *Let $\gamma = \left(\begin{pmatrix} \pi_i a_i & b_i \\ c_i \pi_i^{s_i} & d_i \end{pmatrix} \right) \in U$ and $(\kappa_\psi, r, n, v, w)$ a weight tuple with (κ_ψ, r) an arithmetic weight. Then matrix for the weight $(\kappa, r)\psi$ action of γ in $L\langle X, Y\rangle$ is* *such that the* (x, y)*-th entry has* p*-adic valuation at least*

$$b(x) + g(n_1, x_1, y_1)(s_1 - 1) + g(n_2, x_2, y_2)(s_2 - 1)$$

where $x_i, y_i \in \mathbb{Z}_{\geq 0}$ *and* $x = Bi(x_1, x_2)$*,* $y = Bi(y_1, y_2)$ *and*

$$g(n_i, x_i, y_i) = \begin{cases} \infty & \text{if } x_i > n_i \geq y_i, \\ x_i & \text{if } y_i = 0, \\ 0 & \text{if } y_i \geq x_i, \\ x_i - y_i & \text{if } y_i < x_i. \end{cases}$$

(Note that having infinite p*-adic valuation means that the entry of the matrix is zero.)*

Proof. This follows at once from Proposition 7.1.4 together with Corollary 7.1.11 and noting that $g(n_i, x_i, y_i)$ is either ∞ or the first non-zero t for which $\binom{n_i - y_i}{t}\binom{y_i}{x_i - t} \neq 0$. $\square$

7.2 Slopes of U_p operators

In this section, we want to study the structure of the matrix of U_p and understand its slopes. We will begin by recalling some results on Newton polygons of matrices. Then we will change basis so that U_p becomes an infinite block matrix, with blocks having size $h \times h$ and then study these blocks, in particular, the ones lying on the diagonal. Lastly, we will give a criterion (which can be checked in finite time) for when the block upper triangular submatrix $\mathscr{U}_p$ of U_p has slopes given by unions of arithmetic progressions. The importance of $\mathscr{U}_p$ is that, based on computational evidence, we expect it to have the same Newton polygon (and hence slopes) as U_p.

Definition 7.2.1. If K is a local field and $A \in M_n(K)$ is a matrix, then we define the Newton polygon of A to be the Newton polygon of $\det(1 - XA)$, and denote it $\mathrm{NP}(A)$ and we denote its slopes by $\mathcal{S}(A)$.

In particular, we will talk about the Newton polygon of U_p, by which we mean the Newton polygon of the matrix associated to U_p when seen as a linear map on the space of (overconvergent) Hilbert modular forms.

Let us now recall some basic properties of Newton polygons of matrices. More details can be found in [Ked10].

Definition 7.2.2. Let A be a $n \times n$ matrix over a local field K with uniformizer ϖ. Moreover, let $s_1, s_2, \ldots, s_n$ be such that for $i \in \{1, \ldots, n\}$, $s_1 + \cdots + s_i$ is the minimal valuation of an $i \times i$-minor of A. Then the s_i are called the *elementary divisors* of

A and $\sigma_i(A) = \sigma_i = \varpi^{-s_i}$ are the *singular values* of A. Note that, $\sigma_1(A) = |A| = \max_{i,j}\{|A_{i,j}|\}$.

Theorem 7.2.3. *Let A, B be $n \times n$ matrices. Let $P(T) = 1 + \sum_i a_i T^i$ and $Q(T) = 1 + \sum_i c_i T^i$ denote $\det(1 - XA)$ and $\det(1 - X(A+B))$ respectively. Then*

$$|a_i - c_i| \leq |B| \prod_{j=1}^{i-1} \max\{\sigma_j(A), |B|\}.$$

Proof. This is [Ked10, Theorem 4.4.2] translated into the notation of Fredholm determinants instead of characteristic polynomials. □

Corollary 7.2.4. *Let A, B,P,Q be as in Theorem 7.2.3 and let*

$$f(A, B, i) = |B| \prod_{j=1}^{i-1} \max\{\sigma_j(A), |B|\}.$$

If for all $i \in \{1, \ldots, n\}$, $|a_i| > f(A, B, i)$ then $\mathrm{NP}(A + B) = \mathrm{NP}(A)$.

Proof. If $|a_i| > f(A, B, i)$, then by Theorem 7.2.3, we must have $|a_i| = |c_i|$, from which it follows that the Newton polygons must be the same. □

Proposition 7.2.5. *Let $A_1, \ldots, A_m$ be a set of $n \times n$ matrices over a non-archimedean field K. Then*

$$\mathrm{NP}\left(\bigoplus_i A_i\right) = \sum_i \mathrm{NP}(A_i)$$

where on the right 'sum' is given by the Minkowski sum, i.e., $\mathrm{NP}(A) + \mathrm{NP}(A') = \{\vec{a} + \vec{a}' \mid \vec{a} \in \mathrm{NP}(A), \vec{a}' \in \mathrm{NP}(A')\}$.

Proof. This follows by noting that the characteristic polynomial of a direct sum is a product of the characteristic polynomials of the factors together with the fact that the Newton polygon of a product of polynomials is the same as the Minkowski sum of the Newton polygons. □

7.2.6. Recall that under the isomorphism

$$S^{D,\dagger}_{\kappa_\psi,r}(U) \xrightarrow{\sim} \bigoplus_{i=1}^{h} L\langle X, Y\rangle,$$

the matrix of $\mathfrak{U}_p$ is a block $h \times h$ matrix[2], whose (i, j)-block is given by the infinite matrix

[2] Here h is the class number of (D, U).

of the action of $T_{i,j}$. Now, there is a natural basis[3] of $\bigoplus_{i=1}^h L\langle X, Y\rangle$ such that the matrix of $\mathfrak{U}_p$ becomes an infinite block matrix where each block has size $h \times h$. Moreover, since $T_{i,j} = \sum_{\beta\in\Theta(i,j)}(\gamma_\beta u_\beta)_p$, we have that, in the new basis, the (x, y)-block of $\mathfrak{U}_p$ is given by

$$B_{\kappa_\psi,r}(x, y) := (F_{i,j}^{\kappa_\psi,r}(x, y))_{i,j}$$

where

$$F_{i,j}^{\kappa_\psi,r}(x, y) = \sum_{\beta\in\Theta(i,j)} \Omega_{\kappa_\psi,r}((\gamma_\beta u_\beta)_p, x, y)$$

and $i, j \in \{1, \ldots, h\}$.

Definition 7.2.7. We call these $B_{\kappa_\psi,r}(x, y)$ the ***overconvergent blocks*** of U_p on $S_{\kappa_\psi,r}^{D,\dagger}(U)$.

Notation 7.2.8. For later use we denote by $\Omega_{\kappa_\psi,r}^{(p)}(\gamma, x, y) = \Omega_{\kappa_\psi,r}(\gamma, x, y)/p^{b(x)}$. Similarly, let

$$F_{i,j}^{\kappa,r,(p)}(x, y) = \sum_{\beta\in\Theta(i,j)} \Omega_{\kappa_\psi,r}^{(p)}((\gamma_\beta u_\beta)_p, x, y)$$

and $B_{\kappa_\psi,r}^{(p)}(x, y) = (F_{i,j}^{\kappa_\psi,r,(p)}(x, y))_{i,j}$.

We now make some observations about these overconvergent blocks which we will use later to try and understand the behaviour of the slopes.

Notation 7.2.9. Let $x_1, x_2 \in \mathbb{Z}_{\geq 0}$. We let $\tilde{B}_{\kappa_\psi,r}(x_1, x_2)$, $\tilde{F}_{i,j}^{\kappa_\psi,r}(x_1, x_2)$, $\tilde{\Omega}_{\kappa_\psi,r}(\gamma, x_1, x_2)$, *etc* denote $B_{\kappa_\psi,r}(x, x), F_{i,j}^{\kappa_\psi,r}(x, x), \Omega_{\kappa_\psi,r}(\gamma, x, x)$, *etc* where $x = Bi(x_1, x_2)$. This therefore corresponds to the blocks along the diagonal.

Note that, if the matrix of $\mathfrak{U}_p$ (or equivalently U_p) is given by a infinite $h \times h$ block matrix, then $\tilde{B}_{\kappa_\psi,r}(x_1, x_2)$ is the block corresponding to the basis element $X^{x_1}Y^{x_2}$. Moreover, note that the entries of these block matrices are given by functions $\mathbb{Z}^2 \to \mathcal{O}_L$. Our goal is to understand these functions. Specifically, if we think of the coefficients of the characteristic polynomial of $\tilde{B}^{(p)}(x_1, x_2)$ as functions $\mathbb{Z}^2 \to \mathcal{O}_L$ then we would like to know their p-adic valuations for all x_i. In particular, we would like to know if the have they form $p^n \cdot f(x_1, x_2)$ for some function f (taking values in $\mathcal{O}_L^\times$) and some explicit constant n. Knowing this would at once give the slopes of $\tilde{B}^{(p)}(x_1, x_2)$ for all x_i. With this in mind, we will use the fact that for uniformly continuous functions one can obtain such a decomposition after checking finitely many values (cf. Lemma 7.2.19). Our conjecture is that the $\tilde{B}^{(p)}(x_1, x_2)$ are uniformly continuous in x_1, x_2, but unfortunately we cannot at this time prove this (c.f. 7.2.13).[4]

[3] This is given by grouping the basis elements in each copy by degree.

[4] Note that here we are using $\tilde{B}^{(p)}(-, -)$ so we have taken out the factors at p, since otherwise function could not be uniformly continuous.

Definition 7.2.10. A function $f : \mathbb{Z} \to \mathcal{O}_L$ is called uniformly continuous on $\mathbb{Z}$ (viewed as a subspace of $\mathbb{Z}_p$) if for every δ there exists $\varepsilon = \varepsilon_f(\delta)$ independent of x, x_0 such that if $x, x_0 \in \mathbb{Z}$ and $|x - x_0| < \varepsilon$ then $|f(x) - f(x_0)| < \delta$. More generally, a function $f : \mathbb{Z}^n \to \mathcal{O}_L$ is uniformly continuous on $\mathbb{Z}^n$ if for every δ, there exists a $\varepsilon = \varepsilon_f(\delta)$ independent of $\underline{x}, \underline{x_0} \in \mathbb{Z}^n$ such that if $\max_i(|x_i - x_{0,i}|) < \varepsilon$ then $|f(\underline{x}) - f(\underline{x_0})| < \delta$.

In practice, one can use the Mahler basis to 'see' when a continuous function is uniformly continuous, as follows:

Theorem 7.2.11. *A function $f : \mathbb{Z}^n \to \mathcal{O}_L$ is uniformly continuous on $\mathbb{Z}^n$ if and only if $|a_{\underline{m}}|_p \to 0$ as $\sum_{i=1}^n m_i \to \infty$, where $a_{\underline{m}}$ are the Mahler coefficients of f.*

Proof. See [Mah81, Section 12, Theorem 1]. □

Proposition 7.2.12. *The sum and product of uniformly continuous functions is again a uniformly continuous function.*

Proof. See [Mah81, Section 8, Theorem 5]. □

Conjecture 7.2.13. *For fixed n_i (i.e. fixing the algebraic part of the weight) the functions*

$$\tilde{C}_{n_i}(\gamma, x_i, x_i) : \mathbb{Z} \longrightarrow \mathcal{O}_L$$

(from Proposition 7.1.8) are uniformly continuous in x_i (where $\mathbb{Z} \subset \mathbb{Z}_p$ has the p-adic topology) with $\varepsilon_{\tilde{C}}(\delta) < \delta + 1$.

Remark 7.2.14. For the above function we have checked computationally in many cases that the Mahler coefficients tend to zero p-adically, which lead us to formulate such a conjecture.

Notation 7.2.15. Set $\mathbf{t} = \mathbf{t}_L$ to be the order of the torsion subgroup of $\mathcal{O}_L^\times$.

Proposition 7.2.16. *Let $\alpha \in \mathcal{O}_L^\times$ and let $P(n, \alpha) = \alpha^n$. Then the function $P(n, \alpha^{\mathbf{t}})$ is uniformly continuous as a function $\mathbb{Z} \to \mathcal{O}_L^\times$.*

Proof. The proof of this is just as in [Mah81, Chapter 14, Section 6]. □

Corollary 7.2.17. *Assume 7.2.13. Then, for fixed weight and fixed $s_1, s_2 \in \{0, \ldots, \mathbf{t} - 1\}$, the function*

$$\tilde{B}^{(p)}_{\kappa_\psi, r}(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2) : \mathbb{Z}^2 \longrightarrow \mathrm{Mat}_{h\times h}(\mathcal{O}_L)$$

(see 7.2.8) is uniformly continuous in x_1, x_2.

Proof. This follows at once from Proposition 7.2.16 and our assumption since the entries of $\tilde{B}^{(p)}_{\kappa_\psi, r}$ are given by sums of functions that are uniformly continuous. □

Now, as $\tilde{B}^{(p)}_{\kappa_\psi,r}(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)$ is a $h \times h$ matrix with entries given functions $\mathbb{Z}^2 \to \mathcal{O}_L$, Corollary 7.2.17 gives:

Corollary 7.2.18. *Assume 7.2.13. Then, for fixed weight, the coefficients of the characteristic polynomial of*

$$\tilde{B}^{(p)}_{\kappa_\psi,r}(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)$$

are given by uniformly continuous functions $\mathrm{Coef}_i(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)$ *(here we are suppressing the dependence on the weight) which are uniformly continuous in* x_i*.*

Proof. This follows at once from the fact that the entries of the matrix are given by uniformly continuous functions and the coefficients of the characteristic polynomial are given as sums and products of these functions. □

Later, we will compute (in specific cases) the Newton polygon of U_p and we will make some conjectures on the behaviour of the slopes. Proving these conjectures seems out of reach with our current machinery, but we will give a criterion[5] for there to exist a submatrix of U_p whose Newton polygon matches the conjectural Newton polygon of U_p as suggested by our computations. In the case that we are interested in this submatrix will be the block upper triangular submatrix of U_p.

The criterion is based on the following elementary result:

Lemma 7.2.19. *Let* $f : \mathbb{Z} \to \mathcal{O}_L$ *be a uniformly continuous function. If* $\mathrm{val}_p(f(x)) = \mu$ *for all* $x \in 0, \dots, p^{\mathrm{val}_p(\varepsilon_f(\mu+1))}$ *then* $\mathrm{val}_p(f(x)) = \mu$ *for all* $x \in \mathbb{Z}$*, where* ε_f *is as in Definition 7.2.10.*

Proof. Let $T = \mathrm{val}_p(\varepsilon_f(\mu + 1)$. For $x \in \mathbb{Z}$ we write it as $x = x_i + sp^T$ with $x_i \in \{0, \dots, p^T\}$ and for some $s \in \mathbb{Z}$. Now, since f is uniformly continuous, we know that since $|x - x_i|_p \leq p^{-T}$ we must have $|f(x) - f(x_i)|_p \leq p^{-\mu-1}$. But now, since $\mathrm{val}_p(f(x_i)) = \mu$, we must have $\mathrm{val}_p(f(x)) = \mu$. □

Corollary 7.2.20. *Let* $\mathrm{Coef}_i(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)$ *be as in Corollary 7.2.18. Assume that for each pair* $(s_1, s_2) \in \{0, \dots, \mathbf{t} - 1\}^2$ *there is a* $\mu_i(s_1, s_2) = \mu \in \mathbb{Z}$ *such that*

$$\mathrm{val}_p(\mathrm{Coef}_i(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)) = \mu_i(s_1, s_2)$$

for all $x_1, x_2 \in \{0, \dots, p^T\}$ *with* $T = \varepsilon_{\mathrm{Coef}_i}(\mu + 1)$*. Then*

$$\mathrm{val}_p(\mathrm{Coef}_i(s_1 + \mathbf{t}z, s_2 + \mathbf{t}w)) = \mu_i(s_1, s_2)$$

for all $z, w \in \mathbb{Z}$.

[5] Which one can check in finite time (although this might be take a very long time).

Proof. The proof of this is just a natural generalization of the Lemma 7.2.19. $\square$

Remark 7.2.21. Looking closely at Proposition 7.2.16 and Conjecture 7.2.13, one sees that $\varepsilon_{\mathrm{Coef}_i}(\mu) \leq \mu$ (in fact, we suspect that in many cases it is $\leq \lceil \mu/i \rceil$). This then tells us how far we need to check.

Remark 7.2.22. Note that both Lemma 7.2.19 and Corollary 7.2.20 remain valid if we replace $\mathrm{val}_p(-) = \mu$ with $\mathrm{val}_p(-) \geq \mu$, where μ is now the maximum such integer for which this holds for all pairs of s_i.

Corollary 7.2.23. *Assume 7.2.13. Let κ be a fixed arithmetic weight. For $s_1, s_2 \in \{0, \ldots, \mathbf{t}-1\}$, let*

$$\mu_{\max} = \mu_{\max}(s_1, s_2) = \max_{i=0}^{h} \mu_i(s_1, s_2)$$

(with the same notation as above) and let m be the index of Coef corresponding to $\mu_{\max}$. If for fixed s_1, s_2, the slopes of $\tilde{B}^{(p)}_{\kappa_\psi, r}(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2)$ are fixed for all $x_1, x_2 \in \{0, \ldots, p^T\}$ with $T = \varepsilon_{\mathrm{Coef}_m}(\mu_{\max} + 1)$, then the slopes of $\tilde{B}^{(p)}_{\kappa_\psi, r}(s_1 + \mathbf{t}z, s_2 + \mathbf{t}w)$ will be fixed for all $z, w \in \mathbb{Z}$.

Proof. Using Corollary 7.2.20, we see that our assumptions imply that the break points of our Newton polygon are fixed, so the slopes must be fixed. $\square$

Remark 7.2.24. We note that, in the above, we do not require that the non-break points in $\{(i, \mathrm{val}_p(\mathrm{Coef}_i))\}$ have fixed valuation, but only that they lie above the Newton polygon.

The importance of this result, is that it gives us a way to check for (in finite time) what all of the slopes of the blocks appearing along the diagonal of U_p are (for a fixed weight).

7.2.25. Let $\mathrm{Diag}_{Bi}(p, h)$ denote the infinite diagonal matrix with entries given by

$$\underbrace{1, \ldots, 1}_{h}, \underbrace{p, \ldots, p}_{h}, \ldots, \underbrace{p^{b(n)}, \ldots p^{b(n)}}_{h}, \ldots.$$

It then follows from 7.1.12 that the matrix $\mathfrak{U}_p = \mathrm{Diag}_{Bi}(p, h) \cdot \mathfrak{U}_p^{(p)}$ where $\mathfrak{U}_p^{(p)}$ is an infinite $h \times h$- block matrix whose entries are given by $B^{(p)}_{\kappa_\psi, r}(x, y)$. Lastly, we let $\mathscr{U}_p$ denote the block upper triangular sub matrix of $\mathfrak{U}_p$ whose blocks have size $h \times h$ and the (x, y)-block is given by $B_{\kappa_\psi, r}(x, y)$ if $y \geq x$ and zero otherwise. Note that we can again write $\mathscr{U}_p = \mathrm{Diag}_{Bi}(p, h) \mathscr{U}_p^{(p)}$.

Lemma 7.2.26. *Let κ be fixed and assume that for all $s_1, s_2 \in \{0, \ldots, \mathbf{t}-1\}$ Corollary 7.2.23 holds, meaning that for each pair (s_1, s_2) we have a finite set $\mathcal{S}_{\kappa_\psi, r}(s_1, s_2)$ of slopes* *which will be the slopes of* $\tilde{B}^{(p)}_{\kappa_\psi,r}(s_1+\mathbf{t}z, s_1+\mathbf{t}w)$ *for all* $z,w\in\mathbb{Z}$*. Then the slopes of* $\mathscr{U}_p$ *are given by*

$$\mathcal{S}_{\kappa_\psi,r}(\mathscr{U}_p)=\bigcup_{x=0}^{\infty}\left\{\mathcal{S}(\overline{x_1},\overline{x_2})+b(x)\right\},$$

where $b(x)$ *is as in* 7.1.9 *and* $\overline{x_i}=x_i \mod \mathbf{t}$*. Here for a set of rational numbers* S *we let* $S+a=\{s+a\mid s\in S\}$.

Proof. First note that, since $\mathscr{U}_p$ is block upper triangular, its characteristic polynomial only depends on the blocks along the diagonal. Now, the n-th block along the diagonal will be given by $p^{b(n)}\tilde{B}^{(p)}_{\kappa}(x_1,x_2)$ and by assumption, the slopes of $\tilde{B}^{(p)}_{\kappa}(x_1,x_2)$ only depend on $x_i \mod \mathbf{t}$. So, putting this together with the fact that if a matrix A has a set of slopes $\mathcal{S}(A)$ then $\mathcal{S}(p^nA)=n+\mathcal{S}(A)$, we get the required result. □

One important thing to note is that in this case the set of slopes of $\mathscr{U}_p$ will be given by a union of arithmetic progressions and that the infinite set $\mathcal{S}_\kappa(\mathscr{U}_p)$ is 'generated' by a finite sets $\mathcal{S}(s_1,s_2)$ for $s_i\in\{0,\dots,\mathbf{t}-1\}$.

7.2.27. Now not only do we care about the slopes for a fixed weight, but we would like to know what they all are, at least for weights near the boundary of the weight space. Although this is quite a tough problem in general, we can say something about the set $\mathcal{S}_\kappa(\mathscr{U}_p)$ for κ_ψ, r as the algebraic part of the weight varies.

Looking closely at Corollary 7.1.11, we see that for fixed character ψ and fixed x,y we have

$$\Omega_{\kappa_\psi,r}(\gamma,x,y)=(\text{Constant})\cdot\det(\gamma_1)^{v_1}\det(\gamma_2)^{v_2}d_1^{n_1}d_2^{n_2}C_{n_1}(\gamma_1,x_1,y_2)C_{n_2}(\gamma_2,x,y)$$

where the constant depends on ψ. Now, if we use Hida's normalization for U_p (i.e. U_p^0) then the factors $\det(\gamma_1)^{v_1}\det(\gamma_2)^{v_2}$ disappear so we are reduced to looking at the factors

$$W(n_1,n_2)=d_1^{n_1}d_2^{n_2}C_{n_1}(\gamma_1,x_1,y_1)C_{n_2}(\gamma_2,x_2,y_2).$$

Similar to the above, we see that for fixed $w_1,w_2\in\{0,\dots,\mathbf{t}-1\}$ we have that $W(w_1+\mathbf{t}n_1,w_2+\mathbf{t}n_2)$ is uniformly continuous in n_i. Using this, let $\tilde{\mathbf{B}}^{(p)}(x_1,x_2,n_1,n_2)$ denote $\tilde{B}^{(p)}_{\kappa_\psi,r}(x_1,x_2)$ but now viewed as a function of n_i (the algebraic part of the weight) and x_i.

Proposition 7.2.28. *Assume that for each quadruple* $w_1,w_2,s_1,s_2\in\{0,\dots,\mathbf{t}-1\}$ *the slopes of*

$$\tilde{\mathbf{B}}^{(p)}(s_1+\mathbf{t}x_1,s_2+\mathbf{t}x_2,w_1+\mathbf{t}n_1,w_2+\mathbf{t}n_2)$$

are fixed for all $x_1,x_2,n_1,n_2\in\{0,\cdots,p^T\}$ *with* $T=\epsilon_{\mathrm{Coef}_m}(\mu_{\max}+1)$ *(with the notation as in Corollary* 7.2.23*). Then, assuming* 7.2.13*, the slopes of* $\tilde{\mathbf{B}}^{(p)}(x_1,x_2,n_1,n_2)$ *depend only* *on $x_i, n_i \mod \mathbf{t}$ and in particular, $\mathcal{S}_{\kappa_\psi,r}(\mathscr{U}_p)$ depends only on $x_i, n_i \mod \mathbf{t}$ and is generated as in Lemma 7.2.26.*

Proof. Since $\tilde{\mathbf{B}}^{(p)}(s_1+\mathbf{t}x_1, s_2+\mathbf{t}x_2, w_1+\mathbf{t}n_1, w_2+\mathbf{t}n_2)$ is uniformly continuous in x_i, n_i the proof is just a generalization of Corollary 7.2.23 and Lemma 7.2.26. □

Note that depending only on $n_i \mod \mathbf{t}$ is the same as only depending on which component of the weight space we are in. Therefore, Proposition 7.2.28 is saying that if the slopes $\mathscr{U}_p$ are fixed for enough n_i, x_i, then $\mathcal{S}_{\kappa_\psi,r}(\mathscr{U}_p)$ will only depend on the component of the weight space and moreover the slopes will appear as a union of arithmetic progressions, which are generated by a finite input. This is exactly what one would expect the structure of the slopes of U_p to be. In fact, our computations suggest (cf. 8.3.6) that in many cases (but not all) Proposition 7.2.28 holds and that $\mathcal{S}_{\kappa_\psi,r}(U_p) = \mathcal{S}_{\kappa_\psi,r}(\mathscr{U}_p)$.

7.3 Lower bounds for Newton polygons of U_p operators

We will now prove a lower bound on the Newton polygon of the characteristic polynomial of U_p. This is very much inspired by [WXZ14, Theorem A].

Proposition 7.3.1. *Let U be a sufficiently small level and let (κ, r) be any arithmetic weight. Then the Newton Polygon of the action of U_p on*

$$S^{D,\dagger}_{\kappa,r}(U) \cong \bigoplus_{i=1}^{h} L\langle X, Y\rangle$$

lies above the polygon with vertices

$$(0,0), (h,0), (3h,2h), \ldots, \left(\frac{i(i+1)h}{2}, \frac{(i-1)i(i+1)h}{3}\right), \ldots.$$

Proof. We do this by giving a lower bound for the Hodge polygon of the U_p action, which we recall is always below the Newton polygon.

Now recall that the Hodge polygon is given by the lower convex hull of the vertices (i, min_n), where min_n is the minimal p-adic valuation of the determinants of all $n \times n$ minors. Note that it clearly lies below the Newton polygon. Now Corollary 7.1.12 gives that each $h \times h$ block $B_\kappa(x, y)$ is divisible by $p^{b(x)}$ and note that $\{b(n) | n \in \mathbb{Z}_{\geq 0}\} = \{0, 1, 1, 2, 2, 2, 3, 3, 3, 3, \ldots\}$. Using this we can bound the Hodge polygon from below as follows: let $S = \{s_i\} := \{0_h, 1_{2h}, 2_{3h}, 3_{4h}, \ldots, i_{(i+1)h}, \ldots\}$ where i_n means that i appears n times and let $\Sigma_i = \sum_{j \leq i} s_j$. Then from the above it is easy to see that the

Hodge polygon is bounded from below by the convex hull of the points (i, Σ_i). An easy check then shows that this has break-points at the vertices given above.

$\square$

Note that we are not assuming that the weight lies on the boundary of the weight space. In particular, this lower bound holds everywhere on the weight space.

Chapter 8

Explicit computations

In this section we report on the results of some computations of slopes of U_p, for p a split or inert prime. The computations below were done in `Magma` [BCP97] and `Sage` [Sag16].

Notation 8.0.1. In this section we will use convention given in 1.2.8 for our weights. Using this we will denote arbitrary arithmetic weights as κ and if we want to specify the character we will denote them as $[k_1, k_2]\psi$ where $k_i \in \mathbb{Z}_{\geq 2}$ and paritious. We will also denote $v_p(k,r)$ by $v_p(k)$.

Warning 8.0.2. When computing slopes of overconvergent Hilbert modular forms our strategy is to compute a finite matrix $U_p(N,\kappa)$ which is a $N \times N$ approximation to the infinite matrix of U_p acting on weight (κ) overconverget Hilbert modular forms. The fact that U_p is compact means that we can find a function B such that any vertex of $\mathrm{NP}(U_p(B(N),\kappa))$ of valuation less than N, will also be a vertex of $\mathrm{NP}(U_p(M,\kappa))$ for $M \geq B(N)$. So we can guarantee that the approximation slopes are actually slopes of overconvergent Hilbert modular forms. Here B is a function that depends on the ordering of the basis of the matrix. If we use Bi as in 7.1.9 to order the basis, then $\lfloor \frac{b(N)}{h} \rfloor$ bounds[1] $B(N)$ from below, where (as before) $b(N) = \left\lfloor \frac{-1+\sqrt{1+8N}}{2} \right\rfloor$ and h is the class number for (D,U). Throughout this chapter, when we talk about overconvergent slopes, we mean approximated overconvergent slopes.

In the classical case we do not have this problem and all of the slopes we have computed are actually slopes of classical Hilbert modular forms.

[1]This is most likely not the optimal bound.

8.1 Split case

8.1.1 Computations over $\mathbb{Q}(\sqrt{13})$

Let $F = \mathbb{Q}(\sqrt{13})$ and $p = 3$. We will compute the slopes of U_3 on the space of modular forms of $U_0(9)$ for weights near the boundary. We note here that $U_0(9)$ is sufficiently small, which we checked computationally. In this case we find that $h = 12$, where h is the class number of (D, U) with D/F totally definite with $\mathrm{Disc}(D) = 1$ as usual. We let ψ_r be a continuous character of $\mathcal{O}_p^\times$ of conductor 9 such that $\psi_r(\alpha) = N_{F/\mathbb{Q}}(\alpha)^r$ for $\alpha \in \mathcal{O}_F^\times$. In the following table we list the slopes of classical Hilbert modular forms as a pair (s, m) where s is the slope and m is how many times it appears, i.e., its multiplicity (up to this size of matrix), also we have normalized so that $\mathrm{val}_p(p) = 1$. Note that in our setting we have $U_p = U_{\mathfrak{p}_1} U_{\mathfrak{p}_2}$. We also record here the classical slopes of $U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}$.

Operator	Weight	Classical Slopes
U_p	$[2,2]\psi_2$	$(0,1)$, $(1/2,2)$, $(1,6)$, $(3/2,2)$, $(2,1)$
$U_{\mathfrak{p}_1}$	$[2,2]\psi_2$	$(0,3)$, $(1/2,6)$, $(1,3)$
$U_{\mathfrak{p}_2}$	$[2,2]\psi_2$	$(0,3)$, $(1/2,6)$, $(1,3)$
U_p	$[2,4]\psi_2$	$(0,1)$, $(1/2,2)$, $(1,7)$, $(3/2,4)$, $(2,8)$, $(5/2,4)$, $(3,7)$, $(7/2,2)$, $(4,1)$
$U_{\mathfrak{p}_1}$	$[2,4]\psi_2$	$(0,9)$, $(1/2,18)$, $(1,9)$
$U_{\mathfrak{p}_2}$	$[2,4]\psi_2$	$(0,3)$, $(1/2,6)$, $(1,6)$, $(3/2,6)$, $(2,6)$, $(5/2,6)$, $(3,3)$
U_p	$[2,6]\psi_2$	$(0,1)$, $(1/2,2)$, $(1,7)$, $(3/2,4)$, $(2,8)$, $(5/2,4)$, $(3,8)$, $(7/2,4)$, $(4,8)$, $(9/2,4)$, $(5,7)$, $(11/2,2)$, $(6,1)$
$U_{\mathfrak{p}_1}$	$[2,6]\psi_2$	$(0,15)$, $(1/2,30)$, $(1,15)$
$U_{\mathfrak{p}_2}$	$[2,6]\psi_2$	$(0,3)$, $(1/2,6)$, $(1,6)$, $(3/2,6)$, $(2,6)$, $(5/2,6)$, $(3,6)$, $(7/2,6)$, $(4,6)$, $(9/2,6)$, $(5,3)$
U_p	$[2,8]\psi_2$	(0, 1), (1/2, 2), (1, 7), (3/2, 4), (2, 8), (5/2, 4), (3, 8), (7/2, 4), (4, 8), (9/2, 4), (5, 8), (11/2, 4), (6, 8), (13/2, 4), (7, 7), (15/2, 2), (8, 1)
$U_{\mathfrak{p}_1}$	$[2,8]\psi_2$	$(0,21)$, $(1/2,42)$, $(1,21)$
$U_{\mathfrak{p}_2}$	$[2,8]\psi_2$	$(0,3)$, $(1/2,6)$, $(1,6)$, $(3/2,6)$, $(2,6)$, $(5/2,6)$, $(3,6)$, $(7/2,6)$, $(4,6)$, $(9/2,6)$, $(5,6)$, $(11/2,6)$, $(6,6)$, $(13/2,6)$, $(7,3)$
U_p	$[4,4]\psi_2$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,22)$, $(7/2,10)$, $(4,16)$, $(9/2,6)$, $(5,8)$, $(11/2,2)$, $(6,1)$
$U_{\mathfrak{p}_1}$	$[4,4]\psi_2$	$(0,9)$, $(1/2,18)$, $(1,18)$, $(3/2,18)$, $(2,18)$, $(5/2,18)$, $(3,9)$
$U_{\mathfrak{p}_2}$	$[4,4]\psi_2$	$(0,9)$, $(1/2,18)$, $(1,18)$, $(3/2,18)$, $(2,18)$, $(5/2,18)$, $(3,9)$

U_p	$[4,6]\psi_2$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,23)$, $(7/2,12)$, $(4,24)$, $(9/2,12)$, $(5,23)$, $(11/2,10)$, $(6,16)$, $(13/2,6)$, $(7,8)$, $(15/2,2)$, $(8,1)$
$U_{\mathfrak{p}_1}$	$[4,6]\psi_2$	$(0,15)$, $(1/2,30)$, $(1,30)$, $(3/2,30)$, $(2,30)$, $(5/2,30)$, $(3,15)$
$U_{\mathfrak{p}_2}$	$[4,6]\psi_2$	$(0,9)$, $(1/2,18)$, $(1,18)$, $(3/2,18)$, $(2,18)$, $(5/2,18)$, $(3,18)$, $(7/2,18)$, $(4,18)$, $(9/2,18)$, $(5,9)$
U_p	$[3,3]\psi_1$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,14)$, $(5/2,6)$, $(3,8)$, $(7/2,2)$, $(4,1)$
$U_{\mathfrak{p}_1}$	$[3,3]\psi_1$	$(0,6)$, $(1/2,12)$, $(1,12)$, $(3/2,12)$, $(2,6)$
$U_{\mathfrak{p}_2}$	$[3,3]\psi_1$	$(0,6)$, $(1/2,12)$, $(1,12)$, $(3/2,12)$, $(2,6)$
U_p	$[3,5]\psi_1$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,15)$, $(5/2,8)$, $(3,16)$, $(7/2,8)$, $(4,15)$, $(9/2,6)$, $(5,8)$, $(11/2,2)$, $(6,1)$
$U_{\mathfrak{p}_1}$	$[3,5]\psi_1$	$(0,12)$, $(1/2,24)$, $(1,24)$, $(3/2,24)$, $(2,12)$
$U_{\mathfrak{p}_2}$	$[3,5]\psi_1$	$(0,6)$, $(1/2,12)$, $(1,12)$, $(3/2,12)$, $(2,12)$, $(5/2,12)$, $(3,12)$, $(7/2,12)$, $(4,6)$
U_p	$[3,7]\psi_1$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,15)$, $(5/2,8)$, $(3,16)$, $(7/2,8)$, $(4,16)$, $(9/2,8)$, $(5,16)$, $(11/2,8)$, $(6,15)$, $(13/2,6)$, $(7,8)$, $(15/2,2)$, $(8,1)$
$U_{\mathfrak{p}_1}$	$[3,7]\psi_1$	$(0,18)$, $(1/2,36)$, $(1,36)$, $(3/2,36)$, $(2,18)$
$U_{\mathfrak{p}_2}$	$[3,7]\psi_1$	$(0,6)$, $(1/2,12)$, $(1,12)$, $(3/2,12)$, $(2,12)$, $(5/2,12)$, $(3,12)$, $(7/2,12)$, $(4,12)$, $(9/2,12)$, $(5,12)$, $(11/2,12)$, $(6,6)$
U_p	$[3,9]\psi_1$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,15)$, $(5/2,8)$, $(3,16)$, $(7/2,8)$, $(4,16)$, $(9/2,8)$, $(5,16)$, $(11/2,8)$, $(6,16)$, $(13/2,8)$, $(7,16)$, $(15/2,8)$, $(8,15)$, $(17/2,6)$, $(9,8)$, $(19/2,2)$, $(10,1)$
$U_{\mathfrak{p}_1}$	$[3,9]\psi_1$	$(0,24)$, $(1/2,48)$, $(1,48)$, $(3/2,48)$, $(2,24)$
$U_{\mathfrak{p}_2}$	$[3,9]\psi_1$	$(0,6)$, $(1/2,12)$, $(1,12)$, $(3/2,12)$, $(2,12)$, $(5/2,12)$, $(3,12)$, $(7/2,12)$, $(4,12)$, $(9/2,12)$, $(5,12)$, $(11/2,12)$, $(6,12)$, $(13/2,12)$, $(7,12)$, $(15/2,12)$, $(8,6)$

Remark 8.1.2. There is an involution on the Hilbert modular variety which tells us that the U_p slopes in weight $[k_1,k_2]\psi$ will be the same as those in weight $[k_2,k_1]\psi$.

Since our level is sufficiently small, one can show using 7.0.6, that the dimension of the spaces of classical Hilbert modular forms of weight $[k_1,k_2]\psi_i$ and level $U_0(9)$ is $12\cdot(k_1-1)\cdot(k_2-1)$ for $[k_1,k_2]\neq[2,2]$. For weight $[2,2]$ the dimension of the classical space of cusp forms is 11, but since in our notation we are including the constant functions (which, in this case, contributes a 1-dimensional subspace), we get a 12 dimensional space. With this one can easily see that (as long as we order our basis correctly, which is not be the ordering given by Bi) in the table above, the classical slopes U_p in weight $\kappa = [k_1, k_2]\psi$ are given by the slopes of $U_*(N, \kappa)$ for $N = 12 \cdot (k_1 - 1) \cdot (k_2 - 1)$ and $* \in \{p, \mathfrak{p}_1, \mathfrak{p}_2\}$.

We now compute the overconvergent slope approximations for U_p and the same set of weights as in the classical case. Our computations suggest that for a fixed N and ψ, the set of slopes of $U_p(N, \kappa)$ depend only on the component in which κ lies [2]. For this reason, the table below, we only record the size and the slopes.

Matrix	Overconvergent Slopes
$U_p(20 \cdot 12)$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,39)$, $(11/2,20)$, $(6,35)$, $(13/2,10)$, $(7,5)$
$U_p(22 \cdot 12)$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,40)$, $(11/2,22)$, $(6,42)$, $(13/2,14)$, $(7,12)$, $(15/2,2)$, $(8,1)$
$U_p(25 \cdot 12)$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,40)$, $(11/2,22)$, $(6,45)$, $(13/2,20)$, $(7,30)$, $(15/2,8)$, $(8,4)$
$U_p(28 \cdot 12)$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,40)$, $(11/2,22)$, $(6,48)$, $(13/2,26)$, $(7,48)$, $(15/2,14)$, $(8,7)$
$U_p(30 \cdot 12)$	$(0,1)$, $(1/2,2)$, $(1,8)$, $(3/2,6)$, $(2,16)$, $(5/2,10)$, $(3,24)$, $(7/2,14)$, $(4,32)$, $(9/2,18)$, $(5,40)$, $(11/2,22)$, $(6,48)$, $(13/2,26)$, $(7,50)$, $(15/2,18)$, $(8,19)$, $(17/2,4)$, $(9,2)$

Observation 8.1.3. (1) The slopes are appearing in arithmetic progression which is very similar to what we see over $\mathbb{Q}$.

(2) The multiplicities are not the same for each slope and are increasing, which is something that one does not see over $\mathbb{Q}$ (cf. [LWX14, Therem 1.5]).

(3) In the classical slopes above one can observe the Atkin-Lehner involution in action. We know from Section 3.2 that the Atkin-Lehner involution will send a Hilbert modular form of slope α in $S_{k_\psi}(U_0(9))$ to a form of slope

$$\mathrm{val}_p(N_{F/\mathbb{Q}}(p)^{k_0 - 1 - v_p(k)}) - \alpha,$$

in $S_{k_{\psi^{-1}}}(U_0(9))$ where $k_0 = \max(k_1, k_2)$. Now, in our example, ψ and ψ^{-1} are in the same Galois orbit, so the slopes in weight k_ψ and $k_{\psi^{-1}}$ will be the same. From which one can deduce that in the classical slopes above one should be able to pair up

[2] In general, we expect that the factor $\mathrm{val}_p(w(\kappa))$ only scales the slopes linearly.

the slopes appearing in weight $[k_1, k_2]$ so that the slopes add up to

$$\mathrm{val}_p(N_{F/\mathbb{Q}}(p)^{k_0-1-v_p(k)}),$$

which is the case[3]. Moreover, if instead we look at Atkin-Lehner involutions for each $\mathfrak{p}_i$ for $i \in \{1, 2\}$ we can make similar observations in these cases.

8.1.4 Computations over $\mathbb{Q}(\sqrt{17})$

We have also done a similar computation in the case when $F = \mathbb{Q}(\sqrt{17})$, $p = 2$ and level $U_0(8)$, which again is sufficiently small. Here $h = 24$ with ψ and χ appropriate characters of conductor 8. In this case, we again see a similar structure to the set of slopes.

Operator	Weight	Classical slopes
U_p	$[2,2]\psi$	$(0,1)$, $(1/2,4)$, $(1,14)$, $(3/2,4)$, $(2,1)$
$U_{\mathfrak{p}_1}$	$[2,2]\psi$	$(0,4)$, $(1/2,16)$, $(1,4)$
$U_{\mathfrak{p}_2}$	$[2,2]\psi$	$(0,4)$, $(1/2,16)$, $(1,4)$
U_p	$[4,2]\psi$	$(0,1)$, $(1/2,4)$, $(1,15)$, $(3/2,8)$, $(2,16)$, $(5/2,8)$, $(3,15)$, $(7/2,4)$, $(4,1)$
$U_{\mathfrak{p}_1}$	$[4,2]\psi$	$(0,4)$, $(1/2,16)$, $(1,8)$, $(3/2,16)$, $(2,8)$, $(5/2,16)$, $(3,4)$
$U_{\mathfrak{p}_2}$	$[4,2]\psi$	$(0,12)$, $(1/2,48)$, $(1,12)$
U_p	$[4,4]\psi$	$(0,1)$, $(1/2,4)$, $(1,16)$, $(3/2,12)$, $(2,32)$, $(5/2,20)$, $(3,46)$, $(7/2,20)$, $(4,32)$, $(9/2,12)$, $(5,16)$, $(11/2,4)$, $(6,1)$
$U_{\mathfrak{p}_1}$	$[4,4]\psi$	$(0,12)$, $(1/2,48)$, $(1,24)$, $(3/2,48)$, $(2,24)$, $(5/2,48)$, $(3,12)$
$U_{\mathfrak{p}_2}$	$[4,4]\psi$	$(0,12)$, $(1/2,48)$, $(1,24)$, $(3/2,48)$, $(2,24)$, $(5/2,48)$, $(3,12)$
U_p	$[6,2]\psi$	$(0,\ 1)$, $(1/2,\ 4)$, $(1,\ 15)$, $(3/2,\ 8)$, $(2,\ 16)$, $(5/2,\ 8)$, $(3,\ 16)$, $(7/2,\ 8)$, $(4,\ 16)$, $(9/2,\ 8)$, $(5,\ 15)$, $(11/2,\ 4)$,$(6,\ 1)$
$U_{\mathfrak{p}_1}$	$[6,2]\psi$	$(0,4)$, $(1/2,16)$, $(1,8)$, $(3/2,16)$, $(2,8)$, $(5/2,16)$, $(3,8)$, $(7/2,16)$, $(4,8)$, $(9/2,16)$, $(5,4)$
$U_{\mathfrak{p}_2}$	$[6,2]\psi$	$(0,20)$, $(1/2,80)$, $(1,20)$
U_p	$[8,2]\psi$	$(0,\ 1)$, $(1/2,\ 4)$, $(1,\ 15)$, $(3/2,\ 8)$, $(2,\ 16)$, $(5/2,\ 8)$, $(3,\ 16)$, $(7/2,\ 8)$, $(4,\ 16)$, $(9/2,\ 8)$, $(5,\ 16)$, $(11/2,\ 8)$, $(6,\ 16)$, $(13/2,\ 8)$, $(7,\ 15)$, $(15/2,\ 4)$, $(8,\ 1)$
$U_{\mathfrak{p}_1}$	$[8,2]\psi$	$(0,\ 4)$, $(1/2,\ 16)$, $(1,\ 8)$, $(3/2,\ 16)$, $(2,\ 8)$, $(5/2,\ 16)$, $(3,\ 8)$, $(7/2,\ 16)$, $(4,\ 8)$, $(9/2,\ 16)$, $(5,\ 8)$, $(11/2,\ 16)$, $(6,\ 8)$, $(13/2,\ 16)$, $(7,\ 4)$
$U_{\mathfrak{p}_2}$	$[8,2]\psi$	$(0,28)$, $(1/2,112)$, $(1,28)$

[3] The appearance of $\mathrm{val}_p(N_{F/\mathbb{Q}}(p)^{v_p(k)})$ is due to the normalizations of our operators.

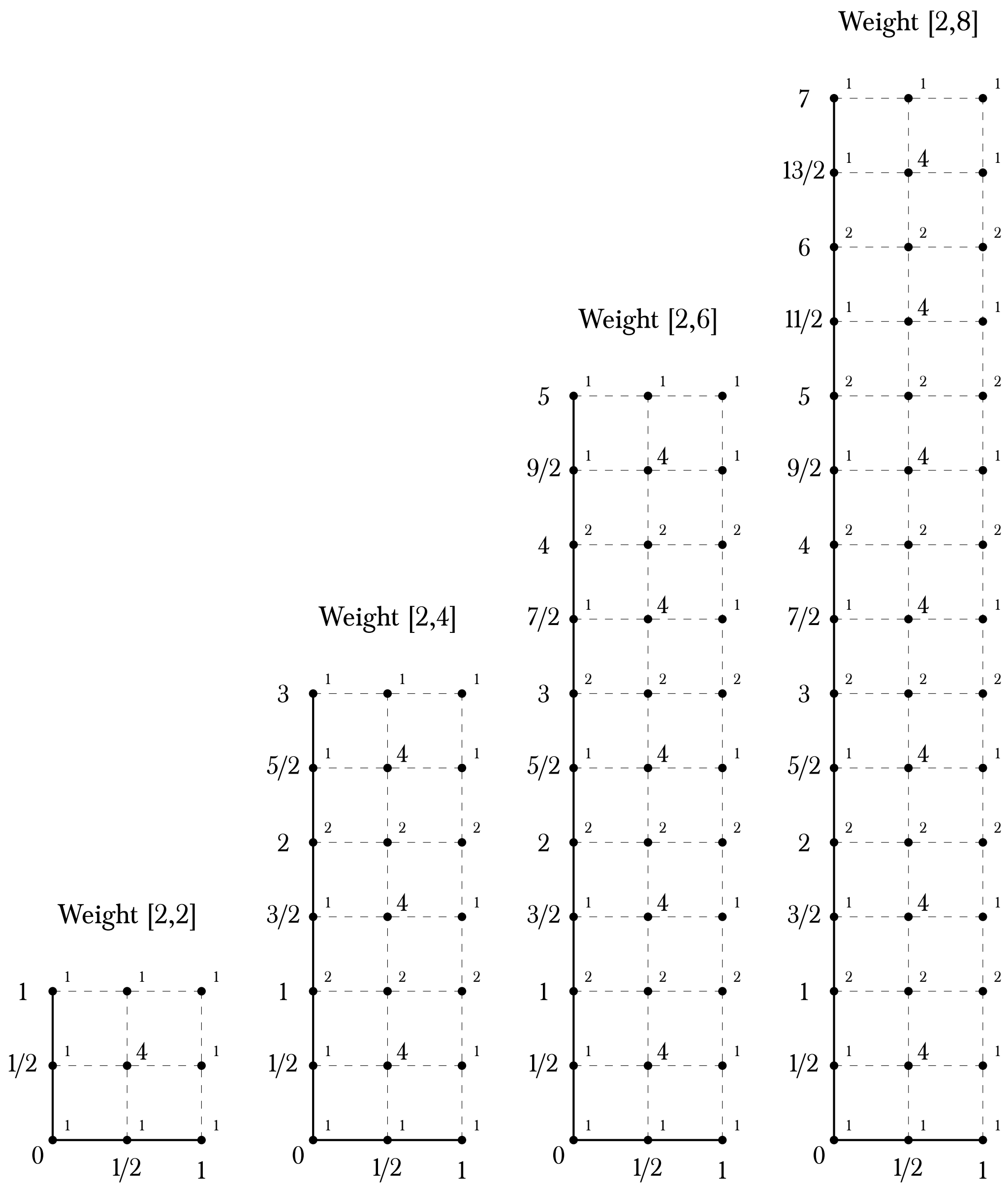
U_p	$[3,3]\chi$	$(0,1)$, $(1/2,4)$, $(1,16)$, $(3/2,12)$, $(2,30)$, $(5/2,12)$, $(3,16)$, $(7/2,4)$, $(4,1)$
$U_{\mathfrak{p}_1}$	$[3,3]\chi$	(0, 8), (1/2, 32), (1, 16), (3/2, 32), (2, 8)
$U_{\mathfrak{p}_2}$	$[3,3]\chi$	(0, 8), (1/2, 32), (1, 16), (3/2, 32), (2, 8)
U_p	$[5,3]\chi$	(0, 1), (1/2, 4), (1, 16), (3/2, 12), (2, 31), (5/2, 16), (3, 32), (7/2, 16), (4, 31), (9/2, 12), (5, 16), (11/2, 4), (6, 1)
$U_{\mathfrak{p}_1}$	$[5,3]\chi$	(0, 8), (1/2, 32), (1, 16), (3/2, 32), (2, 16), (5/2, 32), (3, 16), (7/2, 32), (4, 8)
$U_{\mathfrak{p}_2}$	$[5,3]\chi$	(0, 16), (1/2, 64), (1, 32), (3/2, 64), (2, 16)
U_p	$[7,3]\chi$	(0, 1), (1/2, 4), (1, 16), (3/2, 12), (2, 31), (5/2, 16), (3, 32), (7/2, 16), (4, 32), (9/2, 16), (5, 32), (11/2, 16), (6, 31), (13/2, 12), (7, 16), (15/2, 4), (8, 1)
$U_{\mathfrak{p}_1}$	$[5,3]\chi$	(0, 8), (1/2, 32), (1, 16), (3/2, 32), (2, 16), (5/2, 32), (3, 16), (7/2, 32), (4, 16), (9/2, 32), (5, 16), (11/2, 32), (6, 8)
$U_{\mathfrak{p}_2}$	$[5,3]\chi$	(0, 24), (1/2, 96), (1, 48), (3/2, 96), (2, 24)
Operator	Size	Overconvergent slopes
U_p	$10 \cdot 24$	$(0,1)$, $(1/2,4)$, $(1,16)$, $(3/2,12)$, $(2,32)$, $(5/2,20)$, $(3,48)$, $(7/2,28)$, $(4,59)$, $(9/2,16)$, $(5,4)$
U_p	$20 \cdot 24$	(0, 1), (1/2, 4), (1, 16), (3/2, 12), (2, 32), (5/2, 20), (3, 48), (7/2, 28), (4, 64), (9/2, 36), (5, 79), (11/2, 40), (6, 75), (13/2, 20), (7, 5)

8.1.5 Partial slopes

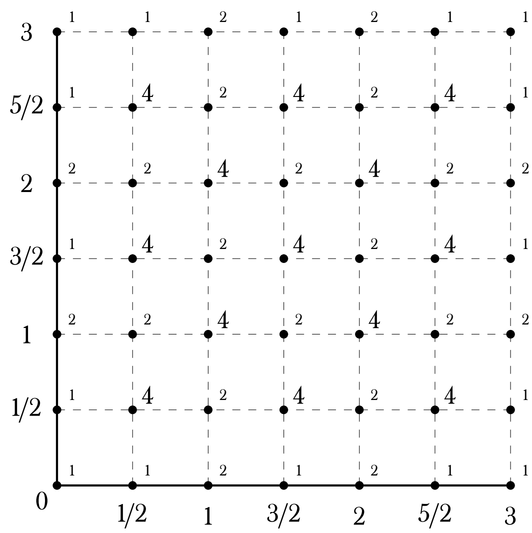
Since we are working in the split case, we have that $U_p = U_{\mathfrak{p}_1}U_{\mathfrak{p}_2} = U_{\mathfrak{p}_2}U_{\mathfrak{p}_1}$ so one can write a U_p slope λ_p as a pair $(\lambda_{\mathfrak{p}_1}, \lambda_{\mathfrak{p}_2})$ where $\lambda_{\mathfrak{p}_i}$ is a slope of $U_{\mathfrak{p}_i}$ and $\lambda_p = \lambda_{\mathfrak{p}_1} + \lambda_{\mathfrak{p}_2}$.

Classical partial slopes

Throughout this subsection, we denote weights $[k_1, k_2]\psi$ simply as $[k_1, k_2]$ with the understanding that we have a character as in the tables above. For level $U_0(9)$ and weights $[2,2], [2,4], [2,6], [2,8], [4,4]$, we plot the pairs $(\lambda_{\mathfrak{p}_1}, \lambda_{\mathfrak{p}_2})$ together with the multiplicity with which they appear.


Weight [2,2]
Weight [2,4]
Weight [2,6]
Weight [2,8]
0
1/2
1
3/2
2
5/2
3
7/2
4
9/2
5
11/2
6
13/2
7
4
2
1

Weight [4,4]



In the above figures the horizontal axis denotes the slopes of $U_{\mathfrak{p}_1}$ and the vertical axis the slopes of $U_{\mathfrak{p}_2}$. The numbers in the grid represent the multiplicity with which this pair appears. Here one sees that if we fix k_1 and let k_2 grow, then the slopes of $U_{\mathfrak{p}_1}$ only increase in multiplicity, but we do not gain any new slopes. On the other hand, for $U_{\mathfrak{p}_2}$ we see that as k_2 increases the we gain new slopes.[4]

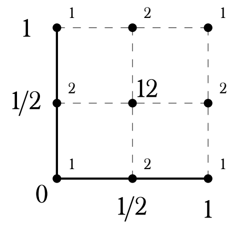
8.1.6. Since we are in the classical case there is no problem in computing the slopes of $U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}$, from which we can construct the above figures as follows: thinking of the multiplicities as variables $(x_{i,j})$, the slopes of $U_{\mathfrak{p}_1}^a U_{\mathfrak{p}_2}^b$ for varying a, b give us linear equations in $(x_{i,j})$ which one can try to solve. For example, knowing that in weight $[2,2]$ the operator $U_{\mathfrak{p}_1}$ has slopes $[(0,3),(1/2,6),(1,3)]$ tells us that in the above figure adding the multiplicities along each column should give $3,6,3$ respectively. Furthermore, the Atkin-Lehner involutions $W_p, W_{\mathfrak{p}_i}$ give extra symmetries in the multiplicities, e.g., W_p sends the pair

$$(\lambda_{\mathfrak{p}_1}, \lambda_{\mathfrak{p}_2}) \longmapsto (k_0 - 1 - \lambda_{\mathfrak{p}_1} - v_{\mathfrak{p}_1}(k), k_0 - 1 - \lambda_{\mathfrak{p}_2} - v_{\mathfrak{p}_2}(k))$$

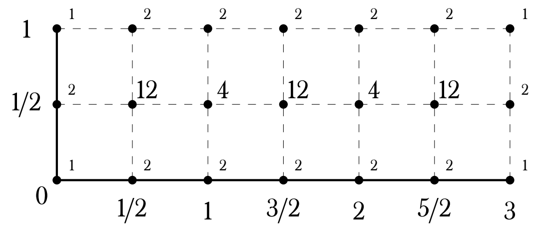
which combined give us enough equations to uniquely determine the multiplicities (for the weights in the above figure).

8.1.7. We draw similar figures in level $U_0(8)$ which give:

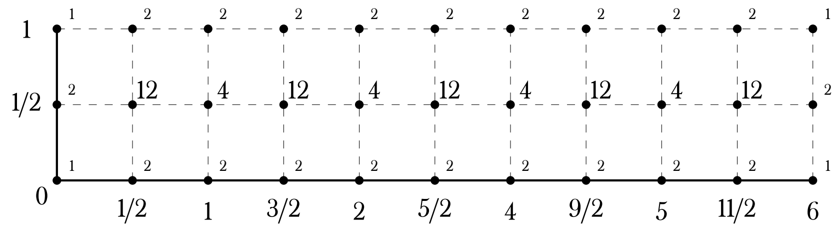
[4]Similarly, if we fix k_2 and increase k_1 we see the same behaviour but with $U_{\mathfrak{p}_1}$ and $U_{\mathfrak{p}_2}$ switching roles.



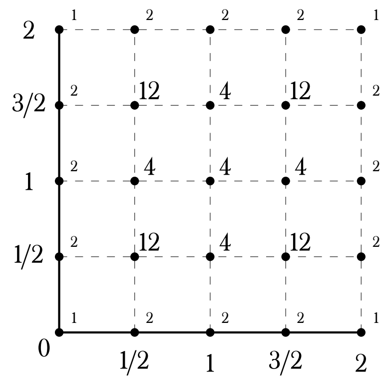
Weight [2,2]



Weight [4,2]



Weight [6,2]



Weight [3,3]

Observation 8.1.8. We note that in both examples above, the pictures appear to built up from the weight $[2,2]$ picture, by 'glueing' along the edges and adding up the multiplicities along the edges.

Question 8.1.9. For (arithmetic) weights near the boundary, are the above multiplicities all ways greater than 0? In other words, given eigenforms f_i for $U_{\mathfrak{p}_i}$ with eigenvalues α_i, does there exist an eigenform for U_p with eigenvalue $\alpha_1\alpha_2\cdots\alpha_f$.

Question 8.1.10. Can we obtain the picture above for any weight near the boundary, by simply glueing the picture in weight $[2,2]$?

Overconvergent partial slopes

8.1.11. In the overconvergent case one cannot directly compute the slopes of $U_{\mathfrak{p}_i}$ since these are not compact operators. Instead one can compute the successive slopes of $U_p U_{\mathfrak{p}_i}^n$ (which are compact operators) for $n \geq 0$. From this one can obtain slopes of $U_{\mathfrak{p}_i}$ as follows: let $N \gg 0$, $P_n(N,\kappa) = (U_p U_{\mathfrak{p}_i}^n)(N,\kappa)$ (with the same notation as in 8.0.2) and let $\mathcal{S}(P_n(N,\kappa))$ denote the set of slopes of $P_n(N,\kappa)$. Now, for each $s \in \mathcal{S}(P_0(N,\kappa))$, let

$$T(s) = \bigcap_{n=1}^{J(s)} \{(t-s)/n \mid t \in \mathcal{S}(P_n(N,\kappa))\}$$

where $J(s) \gg 0$ such that the intersection stabilizes (such a $J(s)$ always exists). Then (for large enough N)

$$\bigcup_{s \in \mathcal{S}(P_0^N(\kappa))} T(s) \subset \mathcal{S}(U_{\mathfrak{p}_i})$$

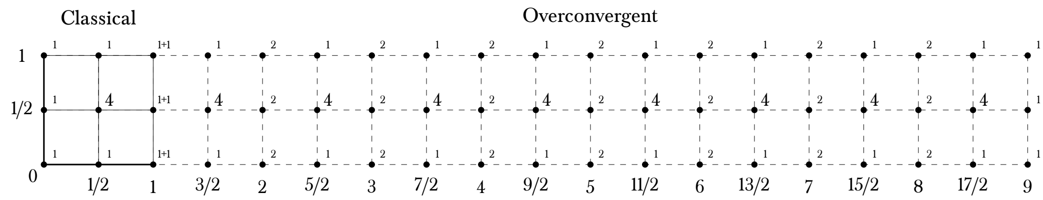
which is what we want.

8.1.12. While $U_{\mathfrak{p}_i}$ are not compact operators on the spaces of overconvergent Hilbert modular forms, one can restrict them to subspaces on which they act as compact operators. To see this, let $L(n,m)$ denote the subspace of $L\langle X,Y\rangle$ generated by X^iY^j for $i \in [0,\dots,n]$ and $j \in [0,\dots,m]$ where $n,m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ (note that $L(\infty,\infty) = L\langle X,Y\rangle$). Then for $m \in \mathbb{Z}_{\geq 0}$, $\kappa = [k_1,k_2]\psi$ a weight with $k_2 = m+2$ and k_1 arbitrary (with appropriate parity conditions), the subspace

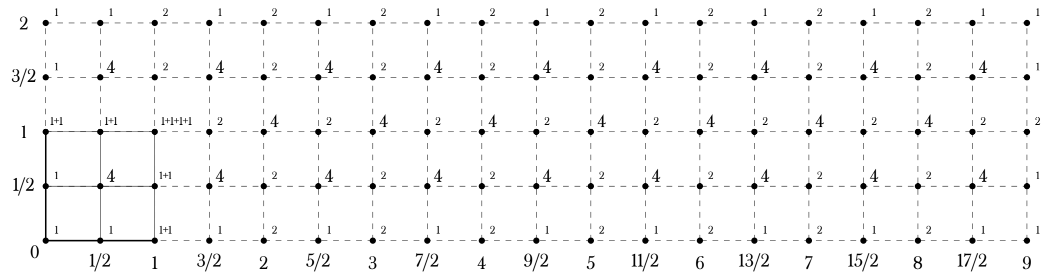
$$\bigoplus_{i=1}^{h} L(\infty,m) \subset \bigoplus_{i=1}^{h} L\langle X,Y\rangle \cong S_\kappa^\dagger(U)$$

is for a fixed under the $|_\kappa$ action of Hecke operators and $U_{\mathfrak{p}_1}$ acts compactly (this can be seen from Corollary 7.1.11) on this subspace. Similarly $U_{\mathfrak{p}_2}$ is compact on $\bigoplus_{i=1}^h L(n,\infty)$ for a fixed $n \in \mathbb{Z}_{\geq 0}$ and weights $\kappa = [n+2,k_2]\psi$. From this one can compute subsets of $\mathcal{S}(U_{\mathfrak{p}_i})$.

Using this we compute some overconvergent slopes of $U_{\mathfrak{p}_i}$ in weight $[2,2]$ acting on $\oplus_i L(0,8) \subset \oplus_i L(0,\infty)$ and $\oplus_i L(8,0) \subset \oplus_i L(\infty,0)$. We only show this for $U_{\mathfrak{p}_1}$ on $\oplus_i L(8,0)$ since the picture for $U_{\mathfrak{p}_2}$ on $\oplus_i L(8,0)$ is the same but flipped vertically. Note that $\oplus_i L(0,0) \cong S_2(U)$.



Slopes of $U_{\mathfrak{p}_1}$ on $\oplus_i L(8,0)$ in weight $[2,2]$



Slopes of $U_{\mathfrak{p}_1}$ on $\oplus_i L(8,1)$ in weight $[2,2]$

The picture for $\oplus_i L(8,1)$ is computed under the assumption that the first 9 multiplicities on the line $y=0$ are as in the picture for $\oplus_i L(8,0)$, which one expects is the case.

Remark 8.1.13. In the computations done in the previous section, it would be interesting to not only vary the k_i independently, but also to choose characters which are more ramified at $\mathfrak{p}_1$ or $\mathfrak{p}_2$ whilst still being in the boundary. This would correspond to moving in the $\mathfrak{p}_1$ or $\mathfrak{p}_2$ 'direction' in the weight space. At the moment we are not able to compute such examples, since in the cases we have studied this would mean increasing the level. This has the effect (in general) of making the matrices much larger, which in turn makes computing the characteristic polynomial much more difficult (which is the bottleneck in our method).

8.2 Inert case

We now move to the inert case. For this we set $F = \mathbb{Q}(\sqrt{5})$ and $p = 2$. We will compute the slopes of U_2 acting on $S^{D,\dagger}_{k_\psi}(U_0(2^3\mathfrak{p}_{11}))$ where $\mathfrak{p}_{11}$ is the prime lying above 11 generated by $(11, 3+2\sqrt{5})$ and $\psi : \mathcal{O}_p^\times \to \{\pm 1\}$ is the primitive Hecke character of conductor 2^3. In particular, it is such that $\psi(e) = 1$ where $e \in \mathcal{O}_F^\times$ is the fundamental unit (embedded in the usual way into $\mathcal{O}_p^\times$) is the fundamental unit. Similarly, we let χ be a primitive Dirichlet character of conductor 2^3 such that $\chi(e) = -1$. Lastly, for $s = [s_1, s_2] \in \mathbb{Z}^2$ let $\tau^s = (\tau_1^{s_1}, \tau_2^{s_2})$ denote the Teichmüller character to the power s. Note that in this case $h = 16$ and therefore the space of classical Hilbert modular forms of weight $[k_1, k_2]\phi$ (for ϕ either of the characters above) and level $U_0(2^3\mathfrak{p}_{11})$ (which can be checked to be sufficiently small) has dimension $(k_1-1)\cdot(k_2-1)16$ for $[k_1,k_2] \neq [2,2]$. In weight $[2,2]\psi$ the dimension is 15, but with our convention, we compute a 16-dimensional space.

8.2.1. Note that ψ is chosen so that we can work with weights with even parity, and χ with odd weights. Moreover, note that in this case $L = \mathbb{Q}_2(\sqrt{5})$ is the degree 2 unramified extension of $\mathbb{Q}_2$. One then checks that the torsion subgroup of the units is cyclic of order 6 given by the 6-th roots of unity. Therefore, an arithmetic weight $[k_1, k_2]\psi$ induces a map on the 6-th roots of unity, and this map determines in what component of the weight space the weight lives. Now looking at the explicit description of the weight, we see that

$$\kappa_\psi(\zeta_6) = \zeta_6^{k_1-k_2}\psi(\zeta_6).$$

From which it follows that for a fixed character ψ, the arithmetic weights given by κ_ψ and κ'_ψ will live on the same component of the weight space if and only if $k_1 - k_2 \equiv k'_1 - k'_2 \mod 6$. Moreover, we can switch between the different components of the weight space by using the Teichmüller character τ.

Weight	Classical Slopes
$[2,2]\psi$	$(2/3,6), (1,4), (4/3,6)$
$[2,2]\psi\tau^2$	$(1/2,4), (1,8), (3/2,4)$
$[2,4]\psi$	$(1/2,4), (1,8), (3/2,4), (5/3,6), (2,4), (7/3,6), (5/2,4), (3,8), (7/2,4)$
$[2,6]\psi$	$(1/2,4), (1,8), (3/2,8), (2,8), (5/2,4), (8/3,6), (3,4), (10/3,6), (7/2,4), (4,8), (9/2,8), (5,8), (11/2,4)$
$[2,8]\psi$	$(2/3,6), (1,4), (4/3,6), (3/2,4), (2,8), (5/2,8), (3,8), (7/2,4), (11/3,6), (4,4), (13/3,6), (9/2,4), (5,8), (11/2,8), (6,8), (13/2,4), (20/3,6), (7,4), (22/3,6)$
$[2,10]\psi$	$(1/2,4), (1,8), (3/2,4), (5/3,6), (2,4), (7/3,6), (5/2,4), (3,8), (7/2,8), (4,8), (9/2,4), (14/3,6), (5,4), (16/3,6), (11/2,4), (6,8), (13/2,8), (7,8), (15/2,4), (23/3,6), (8,4), (25/3,6), (17/2,4), (9,8), (19/2,4)$
$[4,4]\psi$	$(2/3,6), (1,4), (4/3,6), (3/2,8), (2,16), (5/2,16), (8/3,6), (3,20), (10/3,6), (7/2,16), (4,16), (9/2,8), (14/3,6), (5,4), (16/3,6)$
$[4,6]\psi$	$(1/2,4), (1,8), (3/2,8), (5/3,6), (2,12), (7/3,6), (5/2,12), (8/3,6), (3,20), (10/3,6), (7/2,16), (11/3,6), (4,20), (13/3,6), (9/2,16), (14/3,6), (5,20), (16/3,6), (11/2,12), (17/3,6), (6,12), (19/3,6), (13/2,8), (7,8), (15/2,4)$
$[3,3]\chi$	$(2/3,6), (1,4), (4/3,6), (3/2,8), (2,16), (5/2,8), (8/3,6), (3,4), (10/3,6)$
$[3,5]\chi$	$(1/2,4), (1,8), (3/2,8), (5/3,6), (2,12), (7/3,6), (5/2,12), (3,16), (7/2,12), (11/3,6), (4,12), (13/3,6), (9/2,8), (5,8), (11/2,4)$

$[3,7]\chi$	$(1/2,4)$, $(1,8)$, $(3/2,8)$, $(5/3,6)$, $(2,12)$, $(7/3,6)$, $(5/2,8)$, $(8/3,6)$, $(3,12)$, $(10/3,6)$, $(7/2,12)$, $(4,16)$, $(9/2,12)$, $(14/3,6)$, $(5,12)$, $(16/3,6)$, $(11/2,8)$, $(17/3,6)$, $(6,12)$, $(19/3,6)$, $(13/2,8)$, $(7,8)$, $(15/2,4)$
$[3,9]\chi$	$(2/3,6)$, $(1,4)$, $(4/3,6)$, $(3/2,8)$, $(2,16)$, $(5/2,12)$, $(8/3,6)$, $(3,12)$, $(10/3,6)$, $(7/2,8)$, $(11/3,6)$, $(4,12)$, $(13/3,6)$, $(9/2,12)$, $(5,16)$, $(11/2,12)$, $(17/3,6)$, $(6,12)$, $(19/3,6)$, $(13/2,8)$, $(20/3,6)$, $(7,12)$, $(22/3,6)$, $(15/2,12)$, $(8,16)$, $(17/2,8)$, $(26/3,6)$, $(9,4)$, $(28/3,6)$
$[3,11]\chi$	$(1/2,4)$, $(1,8)$, $(3/2,8)$, $(5/3,6)$, $(2,12)$, $(7/3,6)$, $(5/2,12)$, $(3,16)$, $(7/2,12)$, $(11/3,6)$, $(4,12)$, $(13/3,6)$, $(9/2,8)$, $(14/3,6)$, $(5,12)$, $(16/3,6)$, $(11/2,12)$, $(6,16)$, $(13/2,12)$, $(20/3,6)$, $(7,12)$, $(22/3,6)$, $(15/2,8)$, $(23/3,6)$, $(8,12)$, $(25/3,6)$, $(17/2,12)$, $(9,16)$, $(19/2,12)$, $(29/3,6)$, $(10,12)$, $(31/3,6)$, $(21/2,8)$, $(11,8)$, $(23/2,4)$

8.2.2. We now compute some overconvergent slopes, extending our previous computations. As in the split case, the computations suggest that as long as the weights are in the same component of the weight space, they have the same set of slopes. In the table below, we let component 1 consists of the weights $[k_1,k_2]\psi$ appearing the table of classical slopes for which $k_1 \equiv k_2 \mod 6$, and component 2 consist of the remaining weights.

Component	Matrix	Overconvergent Slopes
1	$U_p(20\cdot 16)$	$(2/3,6)$, $(1,4)$, $(4/3,6)$, $(3/2,8)$, $(2,16)$, $(5/2,16)$, $(8/3,6)$, $(3,20)$, $(10/3,6)$, $(7/2,16)$, $(11/3,12)$, $(4,24)$, $(13/3,12)$, $(9/2,24)$, $(14/3,6)$, $(5,36)$, $(16/3,6)$, $(11/2,28)$, $(17/3,12)$, $(6,32)$, $(19/3,12)$, $(13/2,12)$
1	$U_p(22\cdot 16)$	$(2/3,6)$, $(1,4)$, $(4/3,6)$, $(3/2,8)$, $(2,16)$, $(5/2,16)$, $(8/3,6)$, $(3,20)$, $(10/3,6)$, $(7/2,16)$, $(11/3,12)$, $(4,24)$, $(13/3,12)$, $(9/2,24)$, $(14/3,6)$, $(5,36)$, $(16/3,6)$, $(11/2,32)$, $(17/3,12)$, $(6,40)$, $(19/3,12)$, $(13/2,16)$, $(20/3,6)$, $(7,4)$, $(22/3,6)$
1	$U_p(25\cdot 16)$	$(2/3,6)$, $(1,4)$, $(4/3,6)$, $(3/2,8)$, $(2,16)$, $(5/2,16)$, $(8/3,6)$, $(3,20)$, $(10/3,6)$, $(7/2,16)$, $(11/3,12)$, $(4,24)$, $(13/3,12)$, $(9/2,24)$, $(14/3,6)$, $(5,36)$, $(16/3,6)$, $(11/2,32)$, $(17/3,12)$, $(6,40)$, $(19/3,12)$, $(13/2,24)$, $(20/3,12)$, $(7,24)$, $(22/3,12)$, $(15/2,8)$

1	$U_p(28 \cdot 16)$	$(2/3, 6)$, $(1, 4)$, $(4/3, 6)$, $(3/2, 8)$, $(2, 16)$, $(5/2, 16)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 16)$, $(11/3, 12)$, $(4, 24)$, $(13/3, 12)$, $(9/2, 24)$, $(14/3, 6)$, $(5, 36)$, $(16/3, 6)$, $(11/2, 32)$, $(17/3, 12)$, $(6, 40)$, $(19/3, 12)$, $(13/2, 32)$, $(20/3, 18)$, $(7, 44)$, $(22/3, 18)$, $(15/2, 16)$
1	$U_p(30 \cdot 16)$	$(2/3, 6)$, $(1, 4)$, $(4/3, 6)$, $(3/2, 8)$, $(2, 16)$, $(5/2, 16)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 16)$, $(11/3, 12)$, $(4, 24)$, $(13/3, 12)$, $(9/2, 24)$, $(14/3, 6)$, $(5, 36)$, $(16/3, 6)$, $(11/2, 32)$, $(17/3, 12)$, $(6, 40)$, $(19/3, 12)$, $(13/2, 32)$, $(20/3, 18)$, $(7, 44)$, $(22/3, 18)$, $(15/2, 24)$, $(8, 16)$, $(17/2, 8)$
2	$U_p(20 \cdot 16)$	$(1/2, 4)$, $(1, 8)$, $(3/2, 8)$, $(5/3, 6)$, $(2, 12)$, $(7/3, 6)$, $(5/2, 12)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 20)$, $(11/3, 6)$, $(4, 28)$, $(13/3, 6)$, $(9/2, 24)$, $(14/3, 12)$, $(5, 32)$, $(16/3, 12)$, $(11/2, 24)$, $(17/3, 12)$, $(6, 32)$, $(19/3, 12)$, $(13/2, 12)$
2	$U_p(22 \cdot 16)$	$(1/2, 4)$, $(1, 8)$, $(3/2, 8)$, $(5/3, 6)$, $(2, 12)$, $(7/3, 6)$, $(5/2, 12)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 20)$, $(11/3, 6)$, $(4, 28)$, $(13/3, 6)$, $(9/2, 24)$, $(14/3, 12)$, $(5, 32)$, $(16/3, 12)$, $(11/2, 28)$, $(17/3, 12)$, $(6, 40)$, $(19/3, 12)$, $(13/2, 20)$, $(7, 8)$, $(15/2, 4)$
2	$U_p(25 \cdot 16)$	$(1/2, 4)$, $(1, 8)$, $(3/2, 8)$, $(5/3, 6)$, $(2, 12)$, $(7/3, 6)$, $(5/2, 12)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 20)$, $(11/3, 6)$, $(4, 28)$, $(13/3, 6)$, $(9/2, 24)$, $(14/3, 12)$, $(5, 32)$, $(16/3, 12)$, $(11/2, 28)$, $(17/3, 12)$, $(6, 40)$, $(19/3, 12)$, $(13/2, 28)$, $(20/3, 6)$, $(7, 28)$, $(22/3, 6)$, $(15/2, 12)$
2	$U_p(30 \cdot 16)$	$(1/2, 4)$, $(1, 8)$, $(3/2, 8)$, $(5/3, 6)$, $(2, 12)$, $(7/3, 6)$, $(5/2, 12)$, $(8/3, 6)$, $(3, 20)$, $(10/3, 6)$, $(7/2, 20)$, $(11/3, 6)$, $(4, 28)$, $(13/3, 6)$, $(9/2, 24)$, $(14/3, 12)$, $(5, 32)$, $(16/3, 12)$, $(11/2, 28)$, $(17/3, 12)$, $(6, 40)$, $(19/3, 12)$, $(13/2, 36)$, $(20/3, 12)$, $(7, 48)$, $(22/3, 12)$, $(15/2, 24)$, $(23/3, 6)$, $(8, 12)$, $(25/3, 6)$, $(17/2, 4)$

Observation 8.2.3. (1) As in the split case, we see that the slopes are again unions of arithmetic progressions, but the difference is that they do not all have common

difference. In particular, we see arithmetic progressions with common difference $1/2$ and $1/3$ appearing in the sequence of slopes. This again is something that (as far as the author knows) has not been seen over $\mathbb{Q}$, and by results in [LWX14], cannot occur in many cases. Also we have this phenomenon of increasing multiplicities as in the split case.

(2) In the above example, the different components of the weight space are identified by the Galois orbits of the characters. Note that we can move between the components by twisting by the Teichmüller character τ as we did in the weight $[2, 2]$ case.

(3) Again one can see the Atkin-Lehner involution in action in this setting.

8.3 Conjectural behaviour near the boundary

Over $\mathbb{Q}$, [BP16a] have given a conjectural recipe to generate all of the overconvergent slopes and if one looks at this recipe one sees that its only 'ingredients' are classical slopes appearing in weight 2 (with appropriate character) at each component of the weight space and the number of cusps. The analogous behaviour is present in our computations and in general, our computations suggest the following conjectural structure for the slopes.

Conjecture 8.3.1. *Let U be a sufficiently small level and let $\kappa = [k_1, k_2]\psi$ be an arithmetic weight near the boundary. Let $V \in \{U_p, U_{\mathfrak{p}_i}\}$, then for each $r, s \in \{0, \ldots, \mathbf{t} - 1\}$ (with $\mathbf{t}$ the order of the torsion of $\mathcal{O}_L^\times$ as before) there exists a $h \times h$ matrix $B_\kappa(r, s, V)$ which only depends on which component κ lies in (after scaling by $\mathrm{val}_p(w(\kappa))$), such that*

$$\mathcal{S}(V) = \bigcup_{r,s \in \mathbb{Z}_{\geq 0}} \{\mathcal{S}(B_\kappa(\overline{r}, \overline{s}, V)) + r + s\}$$

where $\overline{r}, \overline{s}$ are residues of $r, s \mod \mathbf{t}$. Moreover, on classical subspaces

$$\mathcal{S}(V\mid_{S_\kappa(U)}) = \bigcup_{\substack{r \in \{0 \ldots, k_1 - 2\} \\ s \in \{0, \ldots, k_2 - 2\}}} \{\mathcal{S}(B_\kappa(\overline{r}, \overline{s}, V)) + r + s\}.$$

Remark 8.3.2. Note that, if we identify the classical space $S_\kappa(U)$ with the subspace of $L\langle X, Y\rangle$ with basis $X^r Y^s$ for $r \in \{0, \ldots, k_1 - 2\}, s \in \{0, \ldots, k_2 - 2\}$ then the above conjecture says that associated to each basis element $X^r Y^s$, we have a $h \times h$ matrix $B_\kappa(r, s, X)$, such that if we want to compute the slopes of U_p (or $U_{\mathfrak{p}_i}$) we need only compute the slopes of $B_\kappa(r, s, U_p)$ (or $B_\kappa(r, s, U_{\mathfrak{p}_i})$) for all r, s appearing in the basis of $S_\kappa(U)$. This would then also give a perfect control theorem. Moreover, the matrices

$B_\kappa(r,s,X)$ only depend on $r,s \mod \mathbf{t}$ and on the component of the weight space in which κ lies.

Remark 8.3.3. The conjecture above also explains the fact that the multiplicities of the slopes increasing, suggesting that this is due to the fact that for each $x \in \mathbb{Z}_{\geq 0}$ there are $x+1$ pairs $(x_1,x_2) \in \mathbb{Z}^2_{\geq 0}$ such that $Bi(x_1,x_2) = x$, which in practice means that the generating blocks 'bunch up' giving the increased multiplicities.

8.3.4 Split case

The computations suggest that Conjecture 8.3.1 holds with the following data.

- For $F = \mathbb{Q}(\sqrt{13})$, $U = U_0(9)$, $\kappa = [k_1,k_2]\psi_i$ (as before) and any $r,s \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}\mathcal{S}(B_\kappa(\overline{r},\overline{s},U_p)) =& \{(0,1),(1/2,2),(1,6),(3/2,2),(2,1)\}\\ \mathcal{S}(B_k(\overline{r},\overline{s},U_{\mathfrak{p}_i})) =& \{(0,3),(1/2,6),(1,3)\}\end{aligned}$$

- For $F = \mathbb{Q}(\sqrt{17})$, $U = U_0(8)$, $\kappa = [k_1,k_2]\psi$ or $\kappa = [k_1,k_2]\chi$ (as before) and any $r,s \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned}\mathcal{S}(B_\kappa(\overline{r},\overline{s},U_p)) =& \{(0,1),(1/2,4),(1,14),(3/2,4),(2,1)\}\\ \mathcal{S}(B_k(\overline{r},\overline{s},U_{\mathfrak{p}_i})) =& \{(0,4),(1/2,16),(1,4)\}\end{aligned}$$

Here $\overline{r},\overline{s}$ denote the reduction modulo $\mathbf{t}$ and our computations suggest that the matrices $B_\kappa(\overline{r},\overline{s},X)$ should be the same as the matrices of $U_p, U_{\mathfrak{p}_i}$ acting on classical weight $[2,2]\psi\tau^{r-s}$ where τ is the Teichmüller character, which is analogous to [BP16a, Theorem 3.10].

8.3.5 Inert case

Our computations suggest that Conjecture 8.3.1 holds in the case where $F = \mathbb{Q}(\sqrt{5})$ and $U = U_0(8\mathfrak{p}_{11})$. In this case we have

- Let κ_1 be any arithmetic weight (near the boundary with finite part ψ or χ) in component 1 (see 8.2.2) of the weight space we have

$$\mathcal{S}(B_{\kappa_1}(\overline{r},\overline{s},U_p)) = \begin{cases}\{(2/3,6),(1,4),(4/3,6)\}, & \text{if } r \equiv s \mod 6,\\ \{(1/2,4),(1,8),(3/2,4)\}, & \text{else.}\end{cases}$$

- Let κ_2 an arithmetic weight (near the boundary with finite part ψ or χ) in component 2

$$\mathcal{S}(B_{\kappa_2}(\overline{r},\overline{s},U_p)) = \begin{cases} \{(1/2,4),(1,8),(3/2,4)\}, & \text{if } r \equiv s \mod 6, \\ \{(2/3,6),(1,4),(4/3,6)\}, & \text{else.} \end{cases}$$

8.3.6. Although we cannot at present prove this conjecture, in this case we can try to understand the slopes of $\mathcal{S}_{\kappa,r}(\mathscr{U}_p)$ using Proposition 7.2.28 (assuming 7.2.13). In this case we find that $\mu_{\max} = 16$, $\mathbf{t} = 6$ and therefore (using Remark 7.2.21) one needs to check that for fixed $s_i, w_i \in \{0,\ldots,5\}$, the slopes of

$$\tilde{\mathbf{B}}^{(p)}(s_1 + \mathbf{t}x_1, s_2 + \mathbf{t}x_2, w_1 + \mathbf{t}n_1, w_2 + \mathbf{t}n_2)$$

are fixed for all $x_i, n_i \in \{0,\ldots,2^{17}\}$. We have only checked this for x_1, x_2, k_1, k_2 with $Bi(x_1,x_2) \leq 10^6$ and $0 \leq n_i \leq 10$ (with same parity), since in order completely check this, it would (roughly) take 10^{12} years (on our computers). In all of the cases checked, the slopes agree with those computed in the tables above. This then indicates that the slopes of $\mathcal{S}(\mathscr{U}_p) = \mathcal{S}(U_p)$ and satisfy Conjecture 8.3.1.

Remark 8.3.7. The fact that slopes of $\mathscr{U}_p$ agree with those of U_p appears to be a very general phenomenon of compact operators acting on spaces of convergent power series, as suggested by computing examples of compact operators on $L\langle X, Y\rangle$ which do not correspond to U_p operators. This behaviour is also present for modular forms over $\mathbb{Q}$, as can be seen in [BC05, Jac03].

8.4 The centre of the weight space

To contrast with the computations of slopes near the boundary, we include some computations of slopes near the centre of the weight space. Here we see much less structure than near the boundary.

We now collect some computations of (normalized) slopes for $F = \mathbb{Q}(\sqrt{5})$, $p = 3$ and for weights all in the same component of the weight space, which in this case means $k_1 \equiv k_2 \mod 8$.

Remark 8.4.1. We hope to eventually use this data to construct totally real ghost series analogous to the ones in [BP16b].

Notation 8.4.2. The † denotes overconvergent slopes. So $200^\dagger$ means that these are the first 200 slopes of $U_p(200,\kappa)$ with the notation as in 8.0.2. The other dimensions are the dimension of the corresponding space of classical forms.

Weight	Level	Dimension	Slopes
$[2,2]$	$U_0(\mathfrak{p}_{11})$	0	
$[2,2]$	$U_0(3\mathfrak{p}_{11})$	1	(0, 1), (2, 1)
$[2,2]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 6), (3, 6), (4, 4), (9/2, 4), (5, 8), (11/2, 4), (6, 30), (13/2, 4), (7, 4), (8, 6), (17/2, 4), (9, 6), (10, 22), (11, 10), (23/2, 4), (12, 19), (13, 8), (40/3, 3), (27/2, 6), (14, 30), (29/2, 4), (15, 3), (16, 2)
$[4,4]$	$U_0(\mathfrak{p}_{11})$	1	$(0,1)$
$[4,4]$	$U_0(3\mathfrak{p}_{11})$	18	(0, 1), (2, 16), (6, 1)
$[4,4]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (2, 16), (3, 2), (4, 6), (5, 4), (6, 24), (19/3, 6), (7, 10), (8, 20), (9, 6), (10, 16), (11, 20), (12, 28), (13, 16), (14, 12), (15, 7), (16, 3), (53/3, 3)
$[6,6]$	$U_0(\mathfrak{p}_{11})$	5	(0,1),(1,2),(2,2)
$[6,6]$	$U_0(3\mathfrak{p}_{11})$	50	(0, 1), (1, 2), (2, 2), (4, 40), (8, 2), (9, 2), (10, 1)
$[6,6]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 2), (4, 40), (5, 2), (6, 2), (13/2, 4), (7, 20), (8, 6), (17/2, 4), (9, 10), (19/2, 4), (10, 12), (21/2, 4), (11, 30), (23/2, 4), (12, 17), (25/2, 2), (13, 15), (14, 8), (15, 7), (31/2, 2), (16, 2)
$[8,8]$	$U_0(\mathfrak{p}_{11})$	9	(0,1),(1,2),(2,6)
$[8,8]$	$U_0(3\mathfrak{p}_{11})$	98	(0, 1), (1, 2), (2, 6), (6, 80), (12, 6), (13, 2), (14, 1)
$[8,8]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 6), (6, 80), (7, 2), (8, 2), (9, 26), (10, 2), (21/2, 4), (11, 8), (23/2, 4), (12, 19), (37/3, 3), (13, 21), (27/2, 6), (14, 6), (29/2, 4), (15, 2), (16, 2)
$[10,2]$	$U_0(\mathfrak{p}_{11})$	1	$(0,1)$
$[10,2]$	$U_0(3\mathfrak{p}_{11})$	18	(0, 1), (4, 16), (10, 1)
$[10,2]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 1), (2, 4), (3, 5), (4, 16), (5, 11), (11/2, 4), (6, 5), (13/2, 6), (7, 34), (15/2, 4), (8, 3), (9, 11), (10, 8), (11, 30), (12, 12), (25/2, 4), (38/3, 3), (13, 11), (14, 15), (15, 9), (16, 3)

$[10, 10]$	$U_0(\mathfrak{p}_{11})$	17	(0,1), (2,14), (3,2)
$[10, 10]$	$U_0(3\mathfrak{p}_{11})$	162	(0, 1), (2, 14), (3, 2), (8, 128), (15, 2), (16, 14), (18, 1)
$[10, 10]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (2, 14), (3, 2), (8, 128), (9, 2), (10, 6), (11, 2), (23/2, 4), (12, 8), (37/3, 6), (13, 5), (27/2, 4), (14, 5), (29/2, 2), (15, 5), (16, 3), (17, 2), (18, 1)
$[12, 4]$	$U_0(\mathfrak{p}_{11})$	7	(0,1), (1,1), (2,1), (3,4)
$[12, 4]$	$U_0(3\mathfrak{p}_{11})$	66	(0, 1), (1, 1), (2, 1), (3, 4), (6, 52), (11, 4), (12, 1), (13, 1), (14, 1)
$[12, 4]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 1), (2, 1), (3, 5), (4, 4), (5, 5), (6, 56), (7, 1), (8, 2), (17/2, 12), (9, 18), (19/2, 28), (10, 4), (21/2, 2), (11, 9), (23/2, 4), (12, 8), (25/2, 4), (13, 13), (14, 5), (29/2, 4), (15, 8), (16, 4), (17, 1)
$[14, 6]$	$U_0(\mathfrak{p}_{11})$	13	(0,1), (1,2), (2,2), (3,6), (4,2)
$[14, 6]$	$U_0(3\mathfrak{p}_{11})$	130	(0, 1), (1, 2), (2, 2), (3, 6), (4, 2), (8, 104), (14, 2), (15, 6), (16, 2), (17, 2), (18, 1)
$[14, 6]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 2), (3, 6), (4, 2), (5, 1), (6, 1), (13/2, 2), (7, 3), (15/2, 2), (8, 104), (9, 8), (10, 3), (21/2, 2), (11, 12), (23/2, 14), (47/4, 4), (12, 4), (25/2, 2), (13, 3), (14, 2), (15, 2), (16, 3), (33/2, 2), (17, 6), (18, 7)
$[14, 14]$	$U_0(\mathfrak{p}_{11})$	33	(0,1), (1,2), (2,6), (3,6), (4,4), (5,6), (6,6), (7,2)
$[14, 14]$	$U_0(3\mathfrak{p}_{11})$	338	(0, 1), (1, 2), (2, 6), (3, 6), (4, 4), (5, 6), (6, 6), (7, 2), (12, 272), (19, 2), (20, 6), (21, 6), (22, 4), (23, 6), (24, 6), (25, 2), (26, 1)
$[14, 14]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 6), (3, 6), (4, 4), (5, 6), (6, 6), (7, 2), (12, 58), (13, 1), (27/2, 8), (14, 11), (15, 14), (31/2, 8), (47/3, 6), (16, 10), (33/2, 26), (17, 18), (18, 5), (19, 1), (21, 1)
$[16, 8]$	$U_0(\mathfrak{p}_{11})$	21	(0,1), (1,1), (2,6), (5/2,8), (3,1), (4,4)
$[16, 8]$	$U_0(3\mathfrak{p}_{11})$	210	(0, 1), (1, 1), (2, 6), (5/2, 8), (3, 1), (4, 4), (10, 168), (18, 4), (19, 1), (39/2, 8), (20, 6), (21, 1), (22, 1)

$[16, 8]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 1), (2, 6), (5/2, 8), (3, 1), (4, 4), (7, 1), (8, 2), (9, 3), (10, 111), (12, 1), (13, 2), (27/2, 4), (14, 4), (29/2, 24), (15, 2), (31/2, 2), (16, 7), (17, 2), (18, 2), (37/2, 4), (19, 5), (20, 2), (21, 1)
$[18, 2]$	$U_0(\mathfrak{p}_{11})$	3	(0,1), (1,1), (2,1)
$[18, 2]$	$U_0(3\mathfrak{p}_{11})$	34	(0, 1), (1, 1), (2, 1), (8, 28), (16, 1), (17, 1), (18, 1)
$[18, 2]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 2), (2, 2), (3, 7), (7/2, 4), (4, 3), (9/2, 2), (5, 11), (6, 5), (13/2, 2), (7, 3), (8, 23), (9, 7), (19/2, 6), (10, 8), (21/2, 10), (11, 17), (45/4, 4), (23/2, 8), (12, 12), (25/2, 2), (13, 3), (27/2, 4), (14, 5), (15, 3), (31/2, 2), (16, 10), (33/2, 2), (84/5, 5), (17, 2), (35/2, 2), (18, 5), (37/2, 8), (19, 5), (39/2, 2), (20, 2), (22, 1)
$[20, 4]$	$U_0(\mathfrak{p}_{11})$	11	(0,1), (1,1), (2,5), (3,4)
$[20, 4]$	$U_0(3\mathfrak{p}_{11})$	114	(0, 1), (1, 1), (2, 5), (3, 4), (10, 92), (19, 4), (20, 5), (21, 1), (22, 1)
$[20, 4]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 1), (2, 5), (3, 5), (4, 4), (5, 8), (6, 1), (7, 8), (8, 11), (17/2, 8), (9, 7), (10, 42), (11, 1), (12, 2), (25/2, 2), (13, 5), (40/3, 3), (14, 7), (29/2, 12), (15, 2), (16, 13), (33/2, 2), (17, 8), (35/2, 6), (18, 5), (37/2, 2), (19, 2), (39/2, 6), (41/2, 12), (22, 4), (45/2, 2), (47/2, 2), (24, 1)
$[22, 6]$	$U_0(\mathfrak{p}_{11})$	21	(0,1), (1,1), (2,2), (5/2,8), (3,5), (4,3), (5,1)
$[22, 6]$	$U_0(3\mathfrak{p}_{11})$	210	(0, 1), (1, 1), (2, 2), (5/2, 8), (3, 5), (4, 3), (5, 1), (12, 168), (21, 1), (22, 3), (23, 5), (47/2, 8), (24, 2), (25, 1), (26, 1)
$[22, 6]$	$U_0(3\mathfrak{p}_{11})$	$200^\dagger$	(0, 1), (1, 1), (2, 2), (5/2, 8), (3, 5), (4, 3), (5, 2), (6, 4), (7, 8), (8, 1), (9, 8), (10, 11), (11, 11), (12, 38), (14, 7), (15, 6), (16, 12), (33/2, 16), (17, 12), (18, 10), (19, 4), (20, 8), (21, 7), (22, 3), (23, 3), (24, 5), (25, 1), (26, 3)
$[26, 2]$	$U_0(\mathfrak{p}_{11})$	5	(0,1), (1,1), (2,3)

$[26, 2]$	$U_0(3\mathfrak{p}_{11})$	50	(0, 1), (1, 1), (2, 3), (12, 40), (24, 3), (25, 1), (26, 1)
$[26, 2]$	$U_0(3\mathfrak{p}_{11})$	$200^{\dagger}$	(0, 1), (1, 2), (2, 6), (3, 3), (4, 1), (9/2, 4), (5, 7), (11/2, 6), (6, 28), (13/2, 4), (7, 3), (15/2, 2), (8, 4), (17/2, 2), (9, 11), (10, 3), (11, 6), (12, 8), (13, 3), (27/2, 2), (14, 4), (15, 5), (16, 2), (33/2, 4), (17, 3), (18, 2), (19, 13), (20, 1), (41/2, 2), (21, 9), (22, 2), (67/3, 3), (45/2, 2), (23, 5), (47/2, 2), (24, 5), (25, 5), (51/2, 8), (26, 5), (53/2, 6), (27, 3), (28, 1), (29, 1), (30, 1)

Observation 8.4.3. The first observation is that in this case, the slopes are not appearing as unions of arithmetic progressions. Moreover, there are many non-integer slopes, which is in contrast to many examples over $\mathbb{Q}$.

Remark 8.4.4. In level $U_0(3\mathfrak{p}_{11})$ the dimension of the classical spaces of weight $[k_1, k_2]$ is $2 \cdot (k_1 - 1) \cdot (k_2 - 1)$ since the level is sufficiently small. For level $(U_0(\mathfrak{p}_{11}))$ we have the following table of dimensions, where all the weights are in the same component of the weight space.

Weight	Dim	Weight	Dim	Weight	Dim	Weight	Dim	Weight	Dim
$[2,2]$	0	$[10,2]$	1	$[12,4]$	7	$[14,6]$	13	$[16,8]$	21
$[4,4]$	1	$[18,2]$	3	$[20,4]$	11	$[22,6]$	21	$[24,8]$	33
$[6,6]$	5	$[26,2]$	5	$[28,4]$	17	$[30,6]$	29	$[32,8]$	43
$[8,8]$	9	$[34,2]$	7	$[36,4]$	21	$[38,6]$	37	$[40,8]$	55
$[10,10]$	17	$[42,2]$	9	$[44,4]$	25	$[46,6]$	45	$[48,8]$	65
$[12,12]$	25	$[50,2]$	9	$[52,4]$	31	$[54,6]$	53	$[56,8]$	77
$[14,14]$	33	$[58,2]$	11	$[60,4]$	35	$[62,6]$	61	$[64,8]$	89
$[16,16]$	45	$[66,2]$	13	$[68,4]$	41	$[70,6]$	69	$[72,8]$	99
$[18,18]$	57	$[74,2]$	15	$[76,4]$	45	$[78,6]$	77	$[80,8]$	111
$[20,20]$	73	$[82,2]$	17	$[84,4]$	49	$[86,6]$	85	$[88,8]$	121
$[22,22]$	89	$[90,2]$	17	$[92,4]$	55	$[94,6]$	93	$[96,8]$	133
$[24,24]$	105	$[98,2]$	19	$[100,4]$	59	$[102,6]$	101	$[104,8]$	145
$[26,26]$	125	$[106,2]$	21	$[108,4]$	65	$[110,6]$	109	$[112,8]$	155
$[28,28]$	145	$[114,2]$	23	$[116,4]$	69	$[118,6]$	117	$[120,8]$	167
$[30,30]$	169	$[122,2]$	25	$[124,4]$	73	$[126,6]$	125	$[128,8]$	-
$[32,32]$	193	$[130,2]$	25	$[132,4]$	79	$[134,6]$	133	$[136,8]$	-
$[34,34]$	217	$[138,2]$	27	$[140,4]$	83	$[142,6]$	141	$[144,8]$	-
$[36,36]$	245	$[146,2]$	29	$[148,4]$	89	$[150,6]$	149	$[152,8]$	-

The "-" in the above table indicate that the computation of the dimension had not terminated at the time of writing.

In the table below we work in $\mathbb{Q}(\sqrt{5})$, with $p = 2$ and level $U_0(2\mathfrak{p}_{11})$, where $\mathfrak{p}_{11}|11$.

Level	Weight	Dimension	Slopes
$U_0(\mathfrak{p}_{11})$	$[2,2]$	0	
$U_0(2\mathfrak{p}_{11})$	$[2,2]$	1	$(2,1)$
$U_0(2\mathfrak{p}_{11})$	$[2,2]$	$200^{\dagger}$	$(2,5)$, $(3,2)$, $(4,10)$, $(6,3)$, $(7,4)$, $(8,33)$, $(10,4)$, $(11,2)$, $(12,3)$, $(25/2,4)$, $(13,12)$, $(14,10)$, $(44/3,6)$, $(15,8)$, $(46/3,6)$, $(31/2,8)$, $(16,40)$, $(49/3,3)$, $(33/2,4)$, $(50/3,3)$, $(17,14)$, $(35/2,2)$, $(18,7)$, $(19,3)$, $(20,2)$, $(41/2,2)$
$U_0(\mathfrak{p}_{11})$	$[4,4]$	1	$(2,1)$
$U_0(2\mathfrak{p}_{11})$	$[4,4]$	9	$(2,8)$, $(4,1)$

$U_0(2\mathfrak{p}_{11})$	$[4,4]$	$200^\dagger$	$(2,8)$, $(4,1)$, $(5,4)$, $(16/3,6)$, $(6,6)$, $(7,2)$, $(8,7)$, $(17/2,12)$, $(9,4)$, $(10,21)$, $(11,2)$, $(12,6)$, $(25/2,4)$, $(13,6)$, $(27/2,4)$, $(14,3)$, $(15,22)$, $(16,8)$, $(33/2,4)$, $(17,13)$, $(35/2,24)$, $(18,21)$, $(19,6)$, $(39/2,2)$, $(20,3)$, $(21,1)$
$U_0(\mathfrak{p}_{11})$	$[6,6]$	5	$(2,5)$
$U_0(2\mathfrak{p}_{11})$	$[6,6]$	25	$(2,5),(4,15),(8,5)$
$U_0(2\mathfrak{p}_{11})$	$[6,6]$	$200^\dagger$	$(2,5)$, $(4,15)$, $(6,4)$, $(7,2)$, $(15/2,4)$, $(8,7)$, $(9,4)$, $(10,12)$, $(21/2,20)$, $(11,2)$, $(12,14)$, $(13,4)$, $(14,7)$, $(44/3,6)$, $(15,10)$, $(46/3,6)$, $(16,12)$, $(33/2,4)$, $(17,24)$, $(18,8)$, $(37/2,2)$, $(19,9)$, $(39/2,8)$, $(20,6)$, $(41/2,2)$, $(21,2)$, $(22,1)$
$U_0(\mathfrak{p}_{11})$	$[8,8]$	9	$(2,8),(4,1)$
$U_0(2\mathfrak{p}_{11})$	$[8,8]$	49	$(2,8),(4,1),(6,31),(10,1),(12,8)$
$U_0(2\mathfrak{p}_{11})$	$[8,8]$	$200^\dagger$	$(2,8)$, $(4,1)$, $(6,31)$, $(8,4)$, $(9,2)$, $(19/2,4)$, $(10,3)$, $(11,8)$, $(35/3,6)$, $(12,18)$, $(37/3,6)$, $(25/2,4)$, $(13,4)$, $(14,6)$, $(15,18)$, $(16,7)$, $(17,6)$, $(35/2,12)$, $(18,17)$, $(55/3,6)$, $(37/2,12)$, $(19,9)$, $(20,5)$, $(21,3)$
$U_0(\mathfrak{p}_{11})$	$[4,2]$	1	$(1,1)$
$U_0(2\mathfrak{p}_{11})$	$[4,2]$	3	$(1,2),(3,1)$
$U_0(2\mathfrak{p}_{11})$	$[4,2]$	$200^\dagger$	$(1,2)$, $(2,1)$, $(5/2,2)$, $(3,1)$, $(4,5)$, $(14/3,3)$, $(5,5)$, $(6,1)$, $(13/2,2)$, $(7,5)$, $(8,11)$, $(9,24)$, $(19/2,2)$, $(10,1)$, $(11,9)$, $(12,4)$, $(13,2)$, $(14,19)$, $(29/2,2)$, $(15,7)$, $(46/3,3)$, $(16,12)$, $(49/3,6)$, $(33/2,16)$, $(50/3,3)$, $(67/4,4)$, $(17,21)$, $(35/2,6)$, $(18,9)$, $(37/2,4)$, $(19,5)$, $(20,1)$, $(21,2)$
$U_0(\mathfrak{p}_{11})$	$[6,2]$	1	$(1,1)$
$U_0(2\mathfrak{p}_{11})$	$[6,2]$	5	$(1,1),(2,3),(5,1)$

$U_0(2\mathfrak{p}_{11})$	$[6,2]$	$200^\dagger$	$(1,1)$, $(2,3)$, $(3,2)$, $(10/3,3)$, $(4,1)$, $(9/2,4)$, $(5,4)$, $(17/3,3)$, $(6,2)$, $(7,4)$, $(15/2,4)$, $(8,2)$, $(17/2,2)$, $(9,17)$, $(19/2,2)$, $(10,12)$, $(21/2,2)$, $(11,6)$, $(12,12)$, $(13,5)$, $(14,2)$, $(15,26)$, $(31/2,2)$, $(16,7)$, $(33/2,2)$, $(17,18)$, $(35/2,16)$, $(53/3,6)$, $(18,17)$, $(19,7)$, $(39/2,2)$, $(20,4)$
$U_0(\mathfrak{p}_{11})$	$[6,4]$	3	$(1,2),(3,1)$
$U_0(2\mathfrak{p}_{11})$	$[6,4]$	15	$(1,2),(2,1),(3,9),(6,1),(7,2)$
$U_0(2\mathfrak{p}_{11})$	$[6,4]$	$200^\dagger$	$(1,2)$, $(2,1)$, $(3,9)$, $(4,1)$, $(5,3)$, $(6,4)$, $(19/3,3)$, $(13/2,2)$, $(7,3)$, $(8,8)$, $(9,5)$, $(19/2,16)$, $(10,8)$, $(11,12)$, $(12,3)$, $(13,10)$, $(27/2,2)$, $(41/3,3)$, $(14,7)$, $(43/3,3)$, $(29/2,2)$, $(15,5)$, $(16,28)$, $(17,8)$, $(52/3,3)$, $(35/2,4)$, $(18,12)$, $(37/2,18)$, $(19,7)$, $(20,8)$
$U_0(\mathfrak{p}_{11})$	$[8,4]$	5	$(1,1),(3/2,2),(3,2)$
$U_0(2\mathfrak{p}_{11})$	$[8,4]$	21	$(1,1),(3/2,2),(3,2),(4,11),(7,2),(17/2,2),(9,1)$
$U_0(2\mathfrak{p}_{11})$	$[8,4]$	$200^\dagger$	$(1,1)$, $(3/2,2)$, $(3,2)$, $(4,13)$, $(5,1)$, $(6,9)$, $(7,3)$, $(8,2)$, $(17/2,4)$, $(9,5)$, $(19/2,2)$, $(10,11)$, $(31/3,3)$, $(21/2,4)$, $(32/3,9)$, $(11,7)$, $(35/3,3)$, $(12,5)$, $(13,4)$, $(40/3,3)$, $(27/2,2)$, $(14,4)$, $(43/3,3)$, $(29/2,4)$, $(15,9)$, $(47/3,3)$, $(16,11)$, $(33/2,8)$, $(17,23)$, $(52/3,3)$, $(18,9)$, $(37/2,2)$, $(19,7)$, $(39/2,4)$, $(20,10)$, $(21,2)$, $(22,3)$
$U_0(\mathfrak{p}_{11})$	$[8,6]$	7	$(1,2)$, $(2,1)$, $(5/2,2)$, $(3,1)$, $(4,1)$
$U_0(2\mathfrak{p}_{11})$	$[8,6]$	35	$(1,2)$, $(2,1)$, $(5/2,2)$, $(3,1)$, $(4,1)$, $(5,21)$, $(8,1)$, $(9,1)$, $(19/2,2)$, $(10,1)$, $(11,2)$
$U_0(2\mathfrak{p}_{11})$	$[8,6]$	$200^\dagger$	$(1,2)$, $(2,1)$, $(5/2,2)$, $(3,1)$, $(4,1)$, $(5,21)$, $(7,8)$, $(8,1)$, $(9,7)$, $(19/2,2)$, $(10,4)$, $(31/3,3)$, $(11,15)$, $(34/3,9)$, $(23/2,6)$, $(35/3,3)$, $(12,3)$, $(25/2,2)$, $(13,7)$, $(14,4)$, $(29/2,6)$, $(15,3)$, $(46/3,3)$, $(16,12)$, $(33/2,4)$, $(50/3,3)$, $(17,17)$, $(52/3,3)$, $(35/2,12)$, $(18,13)$, $(37/2,2)$, $(19,6)$, $(20,5)$, $(21,5)$, $(43/2,2)$, $(22,1)$, $(23,1)$

Remark 8.4.5. We can again see that in this case there is much less structure to the slopes. In particular, they do not appear to be unions of arithmetic progressions and their structure is not obviously different from the regular case. Moreover, if one make the

naive extension of the definitions of Γ_0-regular and Γ_0-irregular as in [Buz05], then in the above examples $p = 3$ would be regular and $p = 2$ would be regular, but there does not appear to be any difference in the structure of the slopes in these cases.

8.5 Concluding remarks

The computations done in this chapter clearly indicate that near the boundary of the weight space, the slopes of U_p have a very precise structure analogous to what one sees over $\mathbb{Q}$. The task is to now prove that the slopes are as in Conjecture 8.3.1. Over $\mathbb{Q}$ the analogous result can be shown to hold in many cases by the work of Liu-Wan-Xiao [LWX14]. Their work is based on constructing certain integral models for the spaces of overconvergent quaternionic modular forms (over $\mathbb{Q}$) and then obtaining bounds on the Newton polygon of U_p, which (due to what appears to be a numerical coincidence) is sharp at infinitely many points; this then allows them to deduce very strong results about the geometry of the associated eigenvariety. One of the main obstructions to extending their results to the Hilbert modular form case, is that they rely on combining a stronger version of the control theorem which describes the critical slopes together with the action of the Atkin-Lehner involution. In the Hilbert case we do not at present have a control theorem as strong as this. But the computations suggest that the structure of the slopes of U_p is not uncommon. In particular, for $\gamma \in \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) U_0(p^n)$ say, the matrix of $|_\gamma$ acting on $\oplus_i L\langle X, Y\rangle$ also appear to have this structure. Therefore one should expect a more general proof to work, which explains the numerical coincidences in [LWX14].

More generally, one would like to understand not only the geometry of these eigenvarieties near the boundary, but also in the centre of the weight space. Over $\mathbb{Q}$, Bergdall-Pollack [BP16b] have constructed ghost series, which predict the slopes of U_p near the boundary *and* in the centre. Specifically, in the centre, their ghost series agrees with conjectural algorithms by Buzzard which generate the slopes in the Γ_0-regular case. The striking thing about the construction of the ghost series is that it only relies on dimensions of spaces of classical forms and on dimensions of subspaces of newforms, yet it appears to perfectly predict the slopes in many cases, over large regions of the weight space. In the Hilbert case, one can hope to do something similar by using the data computed above.

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